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CONTINUED FRACTIONS
AND HYPERBOLIC GEOMETRY

MAIRI WALKER, MMATH

Department of Mathematics and Statistics

The Open University



A thesis submitted for the degree of

Doctor of Philosophy

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ABSTRACT

This thesis uses hyperbolic geometry to study various classes of both real and complex continued fractions. This intuitive approach gives insight into the theory of continued fractions that is not so easy to obtain from traditional algebraic methods. Using it, we provide a more extensive study of both Rosen continued fractions and even-integer continued fractions than in any previous works, yielding new results, and revisiting classical theorems. We also study two types of complex continued fractions, namely Gaussian integer continued fractions and Bianchi continued fractions. As well as providing a more elegant and simple theory of continued fractions, our approach leads to a natural generalisation of continued fractions that has not been explored before.

‘Of course it is happening inside your head, Harry,
but why on earth should that mean that it is not real?’

— Albus Dumbledore¹

This thesis is dedicated to my mother, Dr. Jill Walker, who set the ball rolling fifteen years ago when she showed me her own PhD thesis, and gave me the desire to discover something that nobody else has ever discovered before.

¹ in J. K. Rowling's *Harry Potter and the Deathly Hallows*

DECLARATION

I confirm that the material contained in this thesis is the result of independent work. None of it has previously been submitted for a degree or other qualification to this or any other university or institution.

Material from Chapter 2 and Chapter 3 appears in the following two papers.

[54] Short, I. and Walker, M., Even-integer continued fractions and the Farey tree, *accepted for publication*. ArXiv:1508.01373.

[55] Short, I. and Walker, M., Geodesic Rosen continued fractions, *submitted for publication*. ArXiv:1310.1585v2.

December 24th 2015

Mairi Walker

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Special thanks go to my officemates Rosie, Matthew, David and Vasso, who have been so overwhelmingly generous with their advice, support and time that I don't think I'll ever be able to thank them enough. To Ian Allen too, who has always been there to celebrate with me on the good days, and to wipe away the tears and pour the gin on the bad days.

CONTENTS

1	INTRODUCTION	1
1.1	Preliminaries	2
1.2	Continued fractions and hyperbolic geometry	7
1.3	Outline of thesis	17
2	ROSEN CONTINUED FRACTIONS	21
2.1	Rosen continued fractions and hyperbolic geometry	22
2.2	Geodesic Rosen continued fractions	31
2.3	The number of geodesic Rosen continued fractions	43
2.4	Characterising geodesic Rosen continued fractions	66
2.5	Infinite Rosen continued fractions	81
3	EVEN-INTEGER CONTINUED FRACTIONS	85
3.1	Even-integer continued fractions and hyperbolic geometry	86
3.2	The theory of even-integer continued fractions	96
3.3	Even-integer continued fractions and Diophantine approximation	118
3.4	Rosen continued fractions when $\lambda > 2$	133
4	FURTHER TOPICS ON CONTINUED FRACTIONS & HYPERBOLIC GEOMETRY	137
4.1	Gaussian integer continued fractions	138
4.2	Bianchi continued fractions	154
4.3	Semigroups of Möbius transformations	164
	BIBLIOGRAPHY	179

INTRODUCTION

It is well-known that a continued fraction can be studied using sequences of Möbius transformations, and that by using the action of Möbius transformations on hyperbolic space, continued fractions can be represented geometrically in a number of ways. However, little attempt has been made to use these geometric representations to develop a theory of continued fractions, and there are many classes of continued fractions for which no such geometric treatment has been explored.

This thesis uses hyperbolic geometry to study various classes of both real and complex continued fractions. Our intuitive approach gives us insight into the theory of continued fractions that is not so easy to obtain from traditional algebraic methods. It allows us to provide a more extensive study of Rosen continued fractions and even-integer continued fractions than in any previous works, yielding new results, and revisiting classical theorems. We also use it to study two types of complex continued fractions. As well as providing a more simple and elegant theory of continued fractions, our approach leads to a natural generalisation of continued fractions that has not been explored before.

The remainder of this chapter introduces the background material upon which this thesis is built.

1.1 PRELIMINARIES

We begin by introducing the preliminary material on continued fractions, hyperbolic geometry and graph theory that is used throughout this thesis.

1.1.1 CONTINUED FRACTIONS

A *continued fraction* is an expression of the form

$$b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \dots}},$$

where the coefficients a_i and b_i may be any real or complex numbers with $a_i \neq 0$ for $i = 1, 2, \dots$. This thesis concerns continued fractions of the form

$$b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \dots}}, \tag{1.1}$$

where the coefficients b_i belong to some set of real or complex numbers. Such a continued fraction may be *infinite* or *finite*, depending on whether the sequence b_i is infinite or finite respectively. We denote the continued fraction by $[b_1, b_2, \dots]$ in the former case, and by $[b_1, \dots, b_n]$ in the latter.

A *regular continued fraction* is a continued fraction in which all the coefficients b_i are positive integers, except b_1 which may be zero or a negative integer. Although regular continued fractions are the focus of the classical theory of continued fractions, many other classes of continued fractions have been studied extensively; several of these

are discussed in this thesis, as well as some other classes of continued fractions that have received relatively little attention.

The *value* of a finite continued fraction $[b_1, \dots, b_n]$ is the number obtained by evaluating that expression. Sometimes we abuse notation and use $[b_1, \dots, b_n]$ to represent the value of $[b_1, \dots, b_n]$. This is quite natural; the distinction between continued fractions and their values is blurred in most works on continued fractions. The *convergents* of a finite or infinite continued fraction are the continued fractions

$$c_k = [b_1, \dots, b_k],$$

for $k = 1, 2, \dots$. Again, we will sometimes use the term ‘convergent’ to refer to the value of a convergent. If the sequence of convergents of an infinite continued fraction converges to a point x in $\mathbb{C} \cup \{\infty\}$, then we say that the continued fraction *converges* and has *value* x . If the continued fraction $[b_1, b_2, \dots]$ has value x , then we also say that it is a *continued fraction expansion* of x .

We will often find it more convenient to work with ‘minus’ continued fractions of the form

$$b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}. \quad (1.2)$$

rather than the ‘plus’ continued fractions of the form shown in Equation (1.1). In a sense, the two types of continued fractions are equivalent, because we can move from one to the other by changing the sign of the coefficients b_i when i is even. We use the same notation and terminology for minus continued fractions (such as $[b_1, b_2, \dots]$ and so forth) as we do for plus continued fractions.

1.1.2 HYPERBOLIC GEOMETRY

Throughout this thesis we will, unless otherwise specified, use the *upper half-plane* model of two-dimensional hyperbolic space, \mathbb{H} , which is the set

$$\mathbb{H} = \{z = x + yi \in \mathbb{C} \mid y > 0\},$$

equipped with the infinitesimal metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

We will often consider \mathbb{H} along with its ideal boundary $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, the extended real line. Geodesics in \mathbb{H} are either segments of lines $x = c$, where c is a real number, or are arcs of semicircles that are perpendicular to $\hat{\mathbb{R}}$.

The group of conformal isometries of \mathbb{H} , $\text{Isom}^+(\mathbb{H})$, is the group of Möbius transformations

$$\left\{ f(z) = \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} \cong \text{PSL}(2, \mathbb{R}),$$

which also acts as a group of conformal automorphisms of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Non-trivial elements of $\text{Isom}^+(\mathbb{H})$ are *elliptic*, *parabolic* or *hyperbolic* depending on whether the square of the trace $(a + d)^2$ is less than 4, equal to 4, or greater than 4 respectively.

We are particularly interested in discrete subgroups of $\text{Isom}^+(\mathbb{H})$, known as *Fuchsian groups*. A Fuchsian group G acts properly discontinuously on \mathbb{H} , and hence the orbit Gz of a point $z \in \mathbb{H}$ under the action of G has no accumulation points in \mathbb{H} . It may, however, have accumulation points in $\hat{\mathbb{R}}$. We define the *limit set* of G , $\Lambda(G)$, to be

the set of accumulation points of the orbit Gz of a point $z \in \mathbb{H}$ in $\hat{\mathbb{R}}$; the definition is independent of the choice of z . The limit set may be empty, consist of one or two points, or consist of infinitely many points. It is this latter case, in which G is *non-elementary*, that we are interested in, and such a group G can be classified either as a Fuchsian group of the *first kind*, if $\Lambda(G) = \hat{\mathbb{R}}$, or of the *second kind* otherwise. In the latter case, $\Lambda(G)$ is perfect and nowhere dense, or, in other words, $\Lambda(G)$ is a Cantor set. For a Fuchsian group G , we call the set of all fixed points of elliptic (parabolic, hyperbolic) elements of G the set of elliptic (parabolic, hyperbolic) fixed points of G . The limit set $\Lambda(G)$ of G can also be characterised as the closure of the set of non-elliptic fixed points of G .

In Chapter 4 we will work in three-dimensional hyperbolic space. We will use the *upper half-space* model of three-dimensional hyperbolic space, \mathbb{H}^3 , which is the set

$$\mathbb{H}^3 = \{z + tj \mid z \in \mathbb{C}, t \in \mathbb{R}, t > 0\},$$

equipped with the infinitesimal metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2}.$$

We will often consider \mathbb{H} along with its ideal boundary $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the extended complex plane. We identify $\hat{\mathbb{C}}$ with the Euclidean plane $t = 0$ in the obvious way. Geodesics in \mathbb{H}^3 are either segments of lines $z = c$, where c is a complex number, or are arcs of semicircles that are perpendicular to $\hat{\mathbb{C}}$.

The action of any Möbius transformation

$$f(z) = \frac{az + b}{cz + d},$$

with a, b, c , and d complex numbers and $ad - bc \neq 0$, on $\hat{\mathbb{C}}$ has a natural extension to a Möbius transformation acting on \mathbb{H}^3 , called the Poincaré extension. The group of conformal isometries of \mathbb{H}^3 , $\text{Isom}^+(\mathbb{H}^3)$ is the group of Möbius transformations

$$\left\{ f(z) = \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} \cong \text{PSL}(2, \mathbb{C}),$$

which is also the group of all conformal automorphisms of $\hat{\mathbb{C}}$. A discrete subgroup of $\text{Isom}^+(\mathbb{H}^3)$ is known as a *Kleinian group*. A Kleinian group G acts properly discontinuously on \mathbb{H}^3 , and hence the orbit Gz of a point $z \in \mathbb{H}$ under the action of G has no accumulation points in \mathbb{H} . We define the *limit set* $\Lambda(G) \subseteq \hat{\mathbb{C}}$ of a Kleinian group G in the same way that we defined the limit set of a Fuchsian group. Again, we say that G is *non-elementary* if $|\Lambda(G)|$ is infinite, and we say that G is of the *first kind* if $\Lambda(G) = \hat{\mathbb{C}}$ and of the *second kind* otherwise.

1.1.3 GRAPH THEORY

Later on, we will often view continued fractions as paths in graphs embedded in hyperbolic space. Here we introduce some relevant notation and terminology from graph theory.

We consider a graph G to be a set V , whose elements are called *vertices*, together with a collection E of ordered pairs of vertices called *edges*. We are concerned with graphs embedded in either $\mathbb{H} \cup \hat{\mathbb{R}}$ or $\mathbb{H}^3 \cup \hat{\mathbb{C}}$, in which vertices are points in the ideal boundary $\hat{\mathbb{R}}$ or $\hat{\mathbb{C}}$ respectively, and edges are represented by hyperbolic geodesics. Two vertices u and v that are connected by an edge in a graph G are said to be *adjacent* or *neighbours*, and we write $u \sim v$. We denote the edge incident to u and v by $\{u, v\}$ (or $\{v, u\}$). A *path* in G is a sequence of vertices v_0, v_1, \dots, v_n , where $v_{i-1} \sim v_i$ for $i = 1, \dots, n$. We represent

this path by $\langle v_0, v_1, \dots, v_n \rangle$. The edges $\{v_{i-1}, v_i\}$ are called the *edges* of the path. We define the *length* of the path to be n (this is one less than the number of vertices; it corresponds to the number of edges of the path). A *subpath* of $\langle v_0, v_1, \dots, v_n \rangle$ is a path of the form $\langle v_i, \dots, v_j \rangle$, where $0 \leq i < j \leq n$. We consider an *automorphism* of G to be a bijective map f on the vertices of G such that two vertices u and v are adjacent if and only if $f(u)$ and $f(v)$ are adjacent.

Generally, the graph G that we are interested in will have infinitely many vertices. In such a case, we define an *infinite path* in G to be an infinite sequence of vertices v_0, v_1, \dots such that $v_{i-1} \sim v_i$ for all $i = 1, 2, \dots$. We write such a path as $\langle v_0, v_1, \dots \rangle$. If the vertices v_i converge in $\hat{\mathbb{R}}$, say to a number x , then we say that the path *converges* to x and is a *path from v_0 to x* .

Often, we will consider a graph G that is the skeleton of a tessellation of $\mathbb{H} \cup \hat{\mathbb{R}}$ by hyperbolic polygons. In this case, we define the *faces* of G to be polygons in the tessellation.

1.2 CONTINUED FRACTIONS AND HYPERBOLIC GEOMETRY

The classical approach to the theory of continued fractions relies heavily on the recurrence relations

$$p_1 = b_1, \quad p_2 = b_2 b_1 + 1, \quad p_n = b_n p_{n-1} + p_{n-2},$$

and

$$q_1 = 1, \quad q_2 = b_2, \quad q_n = b_n q_{n-1} + q_{n-2},$$

where p_k/q_k are the convergents of the continued fraction $[b_1, b_2, \dots]$. This algebraic approach is used by most textbooks on regular continued fractions (see, for example, [19, 23, 27, 39]), and it is also used to

study a wide variety of other classes of continued fractions. Despite this approach being elementary, it lacks a certain intuition. As said by the mathematician M. C. Irwin [25],

"The proofs are not difficult, but they are usually algebraic, and I find that when I read them I have a tendency to lose sight of where they are leading."

In this thesis, we study continued fractions using hyperbolic geometry, which is an approach that is both intuitive and powerful. Here we outline the connection between continued fractions and hyperbolic geometry, and introduce the three key geometrical representations of continued fractions that we will use: Farey graphs, Ford circles, and cutting sequences. Each approach will be illustrated using integer continued fractions as a concrete example.

1.2.1 CONTINUED FRACTIONS AND MÖBIUS TRANSFORMATIONS

In this section we describe how continued fractions can be viewed as sequences of Möbius transformations. We begin by considering the example of integer continued fractions. An *integer continued fraction* is a minus continued fraction $[b_1, b_2, \dots]$ where each $b_i \in \mathbb{Z}$. We will show that any integer continued fraction, either finite or infinite, can be viewed as a sequence of Möbius transformations belonging to the modular group. The *modular group* Γ is the Fuchsian group generated by the maps

$$\tau(z) = z + 1 \quad \text{and} \quad \sigma(z) = \frac{-1}{z}.$$

Elements of Γ are precisely those Möbius transformations

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c , and d are integers and $ad - bc = 1$, and so $\Gamma \cong \text{PSL}(2, \mathbb{Z})$. The group Γ is a Fuchsian group of the first kind, and the generators τ and σ of Γ satisfy the relations $\sigma^2 = (\tau\sigma)^3 = I$, where I is the identity transformation, and all other relations in τ and σ are consequences of these two. It follows that the group Γ is isomorphic as a group to the free product of cyclic groups $C_2 * C_3$.

Consider an integer continued fraction $[b_1, b_2, \dots]$, which may be either finite or infinite. For each i , define the Möbius transformation

$$f_i(z) = \frac{b_i z - 1}{z} = b_i - \frac{1}{z},$$

that is, $f_i = \tau^i \sigma$ using the generators τ and σ of Γ . Then for each k , we can express the convergent c_k as

$$c_k = [b_1, \dots, b_k] = f_1 f_2 \cdots f_k(\infty),$$

where $f_1 \cdots f_k$ represents the composition of the functions f_1, \dots, f_k . Writing

$$F_k = f_1 \cdots f_k,$$

we see that $[b_1, b_2, \dots]$, whether finite or infinite, can be viewed as a sequence of Möbius transformations F_k belonging to the modular group Γ .

More generally, given a class of continued fractions of the form

$$[b_1, \dots, b_n] = b_1 - \frac{1}{b_2 - \cdots - \frac{1}{b_n}},$$

where each b_i belongs to some set B , we may define functions

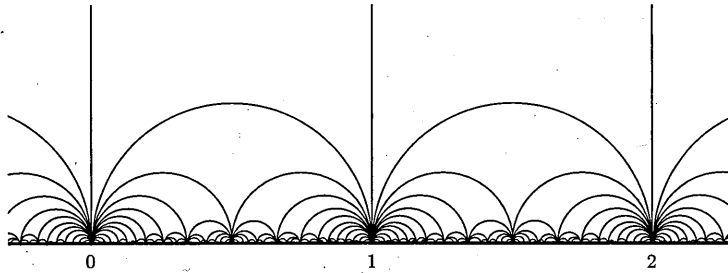
$$f_{b_i}(z) = b_i - 1/z,$$

for each $b_i \in B$. We may then view each continued fraction $[b_1, \dots, b_n]$ as a sequence of Möbius transformations belonging to the group generated by the maps f_b for $b \in B$.

The idea that continued fractions can be viewed as sequences of Möbius transformations can be traced back to the late 19th century, although they were not consistently viewed in this way until the second half of the 20th century [2, 57]. Many of the works that view continued fractions as sequences of Möbius transformations proceed by applying techniques from complex analysis. There are some, however, that make use of the connection between Möbius transformations and hyperbolic geometry. These works lead, in particular, to the three representations of regular continued fractions using hyperbolic geometry that follow.

1.2.2 FAREY GRAPHS

The action of the modular group on \mathbb{H} gives rise to a graph in which certain paths represent integer continued fractions. This graph, which we call the *Farey graph* and denote by \mathcal{F} , may be described as follows. Let δ be the hyperbolic geodesic between 0 and ∞ in \mathbb{H} , along with its endpoints 0 and ∞ . Then the image of δ under elements of the modular group Γ gives rise to a tessellation of \mathbb{H} by ideal hyperbolic triangles. The Farey graph \mathcal{F} is the skeleton of this tessellation, which forms a connected plane graph embedded in $\mathbb{H} \cup \hat{\mathbb{R}}$. The vertices of \mathcal{F} are the vertices of the tessellation, which is the set of all reduced rational numbers, along with the point ∞ , which we identify with $1/0$. The edges of \mathcal{F} are the sides of the ideal triangles in the tessellation; two vertices a/b and c/d of \mathcal{F} are joined by an edge in \mathcal{F} if and only if $|ad - bc| = 1$. The faces of \mathcal{F} are the ideal triangles in the tessellation. A part of \mathcal{F} is shown in Figure 1.1.

Figure 1.1: The Farey graph \mathcal{F}

The Farey graph is so called because of its connection to Farey sequences. The *Farey sequence of order n* is the sequence of reduced rationals between 0 and 1 with denominator at most n , arranged in order of increasing size. Two rational numbers x and y are called *Farey neighbours* if they are neighbouring terms in some Farey sequence. It is well known that if $x = a/b$ and $y = c/d$ then x and y are Farey neighbours if and only if $|ad - bc| = 1$. It follows that the vertices of \mathcal{F} are elements of Farey sequences, and the edges of \mathcal{F} are precisely those hyperbolic geodesics that connect Farey neighbours. The Farey graph appears in numerous other contexts, such as in [56], where it is shown to be a universal cover for any triangular map on a surface, and has been used to study continued fractions in works such as [3] and [48]. The following theorem, which is proved in [3, Theorem 3.1], explains how integer continued fractions may be represented by paths in the Farey graph \mathcal{F} .

Theorem. *Let x be any rational number. Then c_1, \dots, c_n , with $c_n = x$, are the consecutive convergents of some integer continued fraction expansion of x if and only if $\langle \infty, c_1, \dots, c_n \rangle$ is a path in \mathcal{F} from ∞ to x .*

We call the path $\langle \infty, c_1, \dots, c_n \rangle$ the *path of convergents* of x . An example of such a path is illustrated in Figure 1.2.

The correspondence between integer continued fractions and paths in \mathcal{F} is not limited to finite regular continued fractions and finite

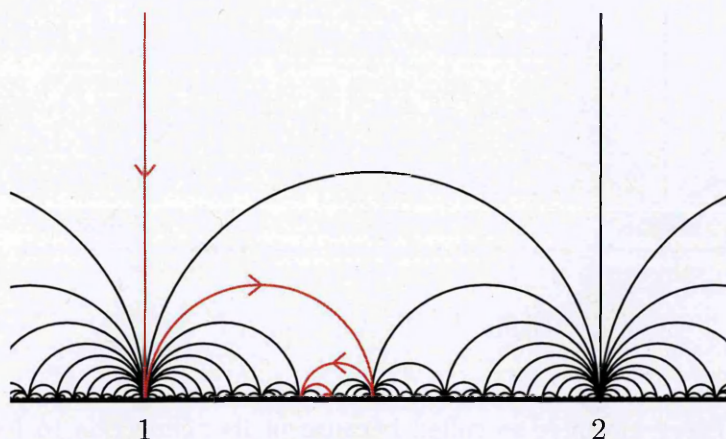


Figure 1.2: A path in \mathcal{F} corresponding to the integer continued fraction of $x = [1, -2, 1, -1, 1]$

paths; the above theorem holds for infinite integer continued fractions and infinite paths too.

There is a simple way to move between an integer continued fraction and its path of convergents; the integers b_1, \dots, b_n of the expansion $[b_1, \dots, b_n]$ of a vertex x encode a set of directions to navigate the corresponding path $\langle \infty, c_1, \dots, c_n \rangle$. In brief, to navigate the path you should, upon reaching c_{i-1} , take the ' b_i th right turn' to get to c_i (which is a left turn if b_i is negative); this can be seen in Figure 1.2. Notice in particular that a regular continued fraction, which is simply an integer continued fraction whose coefficients b_2, b_3, \dots alternate in sign (with b_2 negative), correspond to paths that alternate between left and right turns, and so 'zig-zag' through the graph \mathcal{F} .

We shall see in this thesis that graphs analogous to the Farey graph \mathcal{F} exist in which paths represent other classes of continued fractions. Generally speaking, given a Fuchsian group G that is generated by transformations f_b of the form $f_b(z) = b - 1/z$ where $b \in \mathbb{R}$, we may form a *Farey graph* \mathcal{F}_G as the orbit of the hyperbolic geodesic δ under G . Similarly, a Farey graph embedded in \mathbb{H}^3 may be associated to any Kleinian group that is generated by transformations f_b of the form $f_b(z) = b - 1/z$ where $b \in \mathbb{C}$. In all cases, we obtain a graph in which

paths with initial vertex ∞ represent continued fractions, although it is not always clear what the structure of this graph will be. Some such tessellations appear in works such as [17] and [61], but no study into the corresponding continued fractions has been undertaken.

1.2.3 FORD CIRCLES

In the early 1900s, Lester Ford discovered that regular continued fractions can be viewed as certain chains of tangential horocycles, which we now call Ford circles, in the upper half-plane \mathbb{H} . A *horocycle* is a Euclidean circle in $\mathbb{H} \cup \hat{\mathbb{R}}$ that is tangent to $\hat{\mathbb{R}}$. We say that a horocycle is *based* at an element x of $\hat{\mathbb{R}}$ if the horocycle is tangent to $\hat{\mathbb{R}}$ at x . Given a reduced rational $x = a/b$, the *Ford circle* C_x is the horocycle based at x with Euclidean radius $\text{rad}[C_x] = 1/(2b^2)$. There is one other Ford circle, C_∞ , which is the line $\text{Im}(z) = 1$ in \mathbb{C} together with the point ∞ . A part of the collection of Ford circles is shown in Figure 1.3.

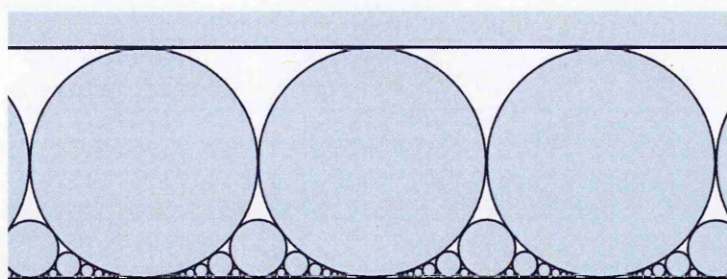


Figure 1.3: Ford circles based at the rational numbers

Two Ford circles are either tangent to one another or completely disjoint. In fact, given $x = a/b$ and $y = c/d$, the Ford circles C_x and C_y are tangent if and only if $|ad - bc| = 1$. It follows that the full collection of Ford circles forms a model for the abstract graph underlying the Farey graph \mathcal{F} ; Ford circles represent the vertices of the graph, and two Ford circles are tangent if and only if their base

points are joined by an edge of \mathcal{F} . The collection of Ford circles shown in Figure 1.3 is shown again in Figure 1.4 with the Farey graph \mathcal{F} superimposed.

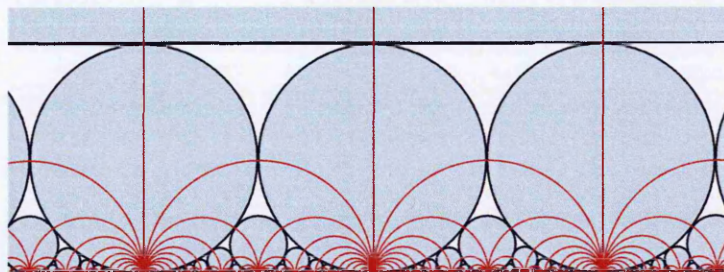


Figure 1.4: Ford circles based at the rational numbers; the Farey graph \mathcal{F} is superimposed

It follows that the Ford circles based at the convergents of an integer continued fraction expansion of a real number x form a chain of tangential Ford circles starting at C_∞ . We call this the *integer continued fraction chain* of x . An example of a part of an integer continued fraction chain is illustrated in Figure 1.5. Note that we may define the *regular continued fraction chain* corresponding to the regular continued fraction expansion of a real number similarly; it is this continued fraction chain that Ford studied in his original papers.

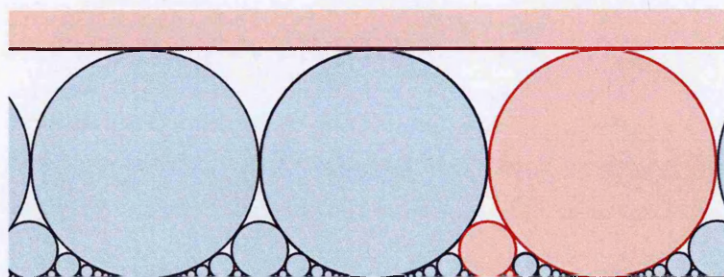


Figure 1.5: The integer continued fraction chain of $x = [2, 2, -2, -2]$

The integer, or regular, continued fraction chain of a real number x encodes the information given by the corresponding path of convergents of x in \mathcal{F} . In addition to this, however, the distance between two convergents, and the distance between a convergent and another rational number, can be computed from the radii of the Ford circles.

This makes Ford circles a useful tool for studying the approximation of irrational numbers by rationals; see, for example, works such as [13], [51] and [53].

In [15], Ford introduced a three-dimensional analogue of Ford circles for complex rational numbers; here, chains of horoballs correspond to continued fractions whose coefficients are Gaussian integers. In [14], Ford uses this collection of horoballs again to study the approximation of complex numbers by complex rationals. In Chapter 3, we introduce a collection of Ford circles that allow us to study the approximation of irrational numbers by numbers lying in a certain subset of the rationals. We also consider a new generalisation of Ford circles.

1.2.4 CUTTING SEQUENCES

We now introduce an entirely different connection between hyperbolic geometry and continued fractions. Taking the quotient of the hyperbolic plane by the modular group gives a surface known as the *modular surface*, which we denote by M . In the 1980s, Richard Moeckel [35] and Caroline Series [52] independently discovered a connection between the symbolic description of the geodesic flow on the modular surface and the theory of regular continued fractions. This connection allows regular continued fractions to be used in the study of the geodesic flow on the modular surface, but it also allows new results on regular continued fractions to be obtained, and classical theorems to be proved in new ways.

Crucial to the connection between the geodesic flow on the modular surface and regular continued fractions is the Farey graph, \mathcal{F} . A directed geodesic ℓ' on the modular surface M lifts to a directed geodesic ℓ in the upper half-plane \mathbb{H} , say with end points x^+ and x^-

such that ℓ is directed towards x^+ . Note that unless ℓ is an edge of the Farey graph \mathcal{F} , we can always choose a lift of ℓ' such that $x^- < 0$ and $x^+ > 0$. Such a geodesic intersects a bi-infinite sequence of faces of \mathcal{F} , as shown in Figure 1.6; we say that ℓ cuts these faces.

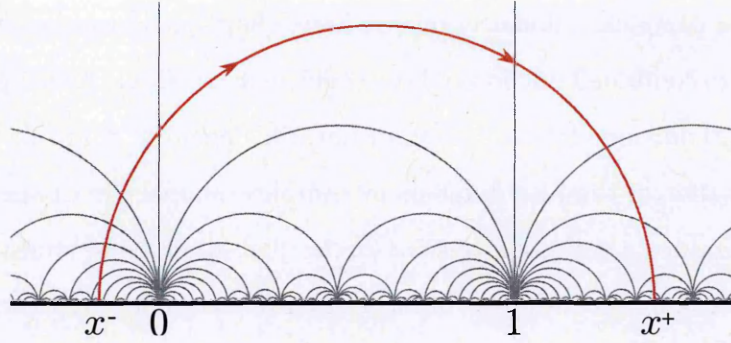


Figure 1.6: A geodesic in \mathbb{H} cuts a bi-infinite sequence of faces of \mathcal{F}

Whenever ℓ cuts a face P of \mathcal{F} , we call that cut a *left cut* (or a *right cut*) if the vertex incident to the two edges of P that ℓ intersects lies to the left (or right) of ℓ . We consider only the sequence of faces P_1, P_2, \dots of \mathcal{F} that ℓ cuts once it passes through the geodesic δ between 0 and ∞ , moving towards x^+ . We may associate to each face P_i the symbol L or R depending on whether it is a left cut or a right cut respectively. Writing LLL as L^3 and so on, we may associate to ℓ the *cutting sequence*

$$L^{b_1} R^{-b_2} L^{b_3} R^{-b_4} \dots,$$

for some sequence of integers b_1, b_2, \dots . The cutting sequence is uniquely determined by x^+ , and it is finite if and only if x^+ is rational. Furthermore, the sequence b_1, b_2, \dots is the sequence of coefficients of the regular continued fraction expansion of x^+ . As an example, the point x^+ shown in Figure 1.6 has a cutting sequence that begins $LR^2LR \dots$. This shows that its regular continued fraction expansion begins $[1, -2, 1, -1, \dots]$.

We can define cutting sequences of geodesics for Farey graphs associated to Fuchsian groups other than the modular group in a similar

way. The cutting sequence with respect to the Farey graphs \mathcal{F}_q described in Chapter 2, for example, gives rise to a Rosen continued fraction expansion of a real number; this has been studied in the Master's thesis of L. Maphakela [34]. The cutting sequence approach has been further adapted by G. Hayward in his Master's thesis [21] to associate to a geodesic in three-dimensional hyperbolic space a continued fraction with Gaussian integer coefficients. In this thesis we use a similar technique to obtain a continued fraction expansion of a complex number with coefficients lying in the set $\mathbb{Z}[\sqrt{2}i]$.

1.3 OUTLINE OF THESIS

In this section we briefly outline the content of this thesis, describing key results and discussing how the work fits in with current literature. Some of the content of this thesis appears in the following two papers.

[54] Short, I. and Walker, M., Even-integer continued fractions and the Farey tree, *accepted for publication*. ArXiv:1508.01373.

[55] Short, I. and Walker, M., Geodesic Rosen continued fractions, *submitted for publication*. ArXiv:1310.1585v2.

The parts of this thesis that appear in [54] and [55] are specified in the following outline.

CHAPTER 2: ROSEN CONTINUED FRACTIONS

In Chapter 2 we study a class of continued fractions that are known as *Rosen continued fractions*. We show how Rosen continued fractions can be viewed as paths in a class of Farey graphs that arise naturally in hyperbolic geometry. This representation gives insight into Rosen's original work about words in Hecke groups, and allows us to introduce the new concept of a *geodesic* Rosen continued fraction. We

describe an algorithm that produces a geodesic Rosen continued fraction expansion of a given real number, and prove that this algorithm is equivalent to the nearest-integer continued fraction algorithm. We provide upper bounds on the number of different geodesic Rosen continued fraction expansions with a given value, and we give a complete characterisation of geodesic Rosen continued fraction expansions in terms of their coefficients.

Much of this research has been submitted for publication in the paper, [55]. Several key results, however, are included in this chapter but are beyond the scope of the paper [55]. In particular, the proof of Theorem 2.18 and Theorem 2.19, and the contents of Section 2.3.3 are unpublished.

CHAPTER 3: EVEN-INTEGER CONTINUED FRACTIONS

In Chapter 3 we study a class of continued fractions, that we call *even-integer continued fractions*, and which are closely related to Rosen continued fractions. We show that even-integer continued fractions can be represented by paths in a Farey graph, and prove that this graph is a tree. It follows that all even-integer continued fraction expansions are geodesic; we turn our attention away from the notion of geodesic continued fractions and towards expounding the theory of even-integer continued fractions.

We prove that almost all real numbers have a unique even-integer continued fraction expansion, and describe an algorithm that produces an even-integer continued fraction expansion of any real number. We prove analogues for even-integer continued fraction of several classical theorems from the theory of regular continued fractions, including Serret's theorem and Lagrange's theorem. We also study Diophantine approximation, using Ford circles to show that even-integer

continued fractions have properties that are analogous to the approximation properties of regular continued fractions. We also find the Hurwitz constant for this approximation.

Many of the results in Chapter 3 are unoriginal. Our method of proof using hyperbolic geometry, however, is original, and it is simple and elegant. Additionally, the results in Section 3.2.6 and Section 3.3.3 are new, as is the discussion in Section 3.4. The proof of Theorem 3.13 is also new in the sense that it is the first proof of the theorem not to assume its counterpart for regular continued fractions. Many of the results in Chapter 3 have been accepted for publication in [54], although we include here the details of many arguments that are omitted from [54], and expand on much of the content. Additionally, the results in Section 3.2.3, Section 3.2.5, and Section 3.3.3 are unpublished, as is the discussion in Section 3.4.

CHAPTER 4: FURTHER TOPICS IN CONTINUED FRACTIONS AND HYPERBOLIC GEOMETRY

In Chapter 4 we look into several topics that extend the work of Chapter 2 and Chapter 3. Much of the research presented in this chapter is in its early stages, and, although some results are obtained, this section aims primarily to point to areas for future research.

Section 4.1 investigates the possibility of adapting the techniques used in Chapter 2 and Chapter 3 to study Gaussian integer continued fractions, the natural analogue of integer continued fractions for complex continued fractions. It is shown that Gaussian integer continued fractions can be represented as paths in a graph embedded in the hyperbolic upper half-space \mathbb{H}^3 . A new condition for the convergence of Gaussian integer continued fractions is obtained, and potential areas for further research are presented. In particular, it is discussed

how the technique introduced in this section might illuminate the recent work of Dani and Nogueira [11] on Gaussian integer continued fractions, and might lead to improvement of the results in [11].

Section 4.2 focuses on a class of continued fractions that we call *Bianchi continued fractions*. There has been little study of Bianchi continued fractions, despite the fact that they are closely related to Gaussian integer continued fractions. Adapting the technique of Moeckel [35] and Series [52], we associate to a directed geodesic in \mathbb{H}^3 a Bianchi continued fraction. This leads to a new Bianchi continued fraction expansion of a complex number. The work in this section is the result of joint research with Meira Hockman from the University of the Witwatersrand, Johannesburg, South Africa.

Section 4.3 takes a very different approach to that taken previously in the thesis. We show that regular continued fractions can be viewed as sequences of elements of a two-generator semigroup of Möbius transformations. We propose the idea of studying certain two-generator semigroups of Möbius transformations as natural generalisations of regular continued fractions, which leads to the concept of *S-continued fractions*. We find that under certain constraints, analogues of both Farey graphs and Ford circles may be constructed that allow the study of *S-continued fractions*. We discuss several directions in which this research might lead. The ideas presented in this section are the result of joint research with Ian Short and Matthew Jacques from The Open University.

ROSEN CONTINUED FRACTIONS

In 1954, David Rosen [41] introduced a class of continued fractions, now known as Rosen continued fractions, in order to study their associated discrete groups of Möbius transformations, the Hecke groups. Since then a rich literature on Rosen continued fractions has developed, including works on Diophantine approximation [32, 41], the metrical theory of Rosen continued fractions [7], dynamics and geometry on surfaces associated to Hecke groups [42], and, most recently, on transcendence results for Rosen continued fractions [6]. In this chapter we describe how Rosen continued fractions can be represented by paths in a class of graphs embedded in \mathbb{H} that are natural analogues of the Farey graph introduced in Section 1.2.2. This perspective gives insight into Rosen's original work, and allows us to identify and study Rosen continued fractions of shortest length.

The structure of this chapter is as follows. In Section 2.1 we define Rosen continued fractions and show how they can be associated to paths in a class of Farey graphs \mathcal{F}_q . In Section 2.2 we introduce the idea of geodesic finite Rosen continued fractions – those finite Rosen continued fractions whose corresponding paths in \mathcal{F}_q are geodesic in the graph metric – and show that such Rosen continued fractions can be obtained using the nearest-integer algorithm. In Section 2.2 we study the possible number of geodesic finite Rosen continued fraction expansions with a given value, providing sharp upper bounds on this quantity. In Section 2.4 we present a way of characterising geodesic finite Rosen continued fractions in terms of their coefficients. Finally,

in Section 2.5 we briefly study geodesic infinite Rosen continued fractions.

2.1 ROSEN CONTINUED FRACTIONS AND HYPERBOLIC GEOMETRY

A *Rosen continued fraction* is a continued fraction of the form

$$b_1\lambda_q + \frac{-1}{b_2\lambda_q + \frac{-1}{b_3\lambda_q + \dots}},$$

where each $b_i \in \mathbb{Z}$, and $\lambda_q = 2\cos\left(\frac{\pi}{q}\right)$ for some integer $q \geq 3$. We denote an infinite Rosen continued fraction by $[b_1, b_2, \dots]_q$, and a finite Rosen continued fraction by $[b_1, \dots, b_n]_q$; this notation is slightly different from that introduced in Section 1.1.1, but allows us to omit λ_q from each coefficient. In this chapter we are concerned mostly with finite Rosen continued fractions, although infinite Rosen continued fractions are studied briefly in Section 2.5. Notice that when $q = 3$, we have $\lambda_q = 1$, and we obtain integer continued fractions. We will often omit this case, as results analogous to those given in this chapter have been obtained for integer continued fractions using similar techniques in [3] (this case is also discussed in Section 1.2).

In this section we will show that Rosen continued fractions can be viewed as paths in certain graphs embedded in \mathbb{H} . These graphs arise naturally from the action of Hecke groups on \mathbb{H} , and so we begin by discussing the Hecke groups and their relation to Rosen continued fractions.

2.1.1 THE HECKE GROUPS

Given a real number $\lambda > 0$, the *Hecke group* $\Gamma(\lambda)$ is the group of Möbius transformations generated by

$$\tau_\lambda(z) = z + \lambda \quad \text{and} \quad \sigma(z) = -\frac{1}{z}.$$

The Hecke groups were introduced in 1936 by Erich Hecke [22], who used them in his study of Dirichlet series. Hecke showed that $\Gamma(\lambda)$ is a Fuchsian group if and only if either $\lambda = 2\cos\left(\frac{\pi}{q}\right)$ for some integer $q \geq 3$ or if $\lambda \geq 2$. In the first case, and when $\lambda = 2$, $\Gamma(\lambda)$ is a Fuchsian group of the first kind; when $\lambda > 2$, $\Gamma(\lambda)$ is a Fuchsian group of the second kind. In this chapter we will consider $\Gamma(\lambda)$ in the case in which $\lambda = \lambda_q = 2\cos\left(\frac{\pi}{q}\right)$; the case $\lambda = 2$ will be considered in Chapter 3, and the cases $\lambda > 2$ will be discussed briefly in Section 3.4.

Given $q \geq 3$, denote by Γ_q the group $\Gamma(\lambda_q)$, and by τ the transformation τ_{λ_q} wherever this will not cause ambiguity. The generators σ and τ of Γ_q satisfy the relations $\sigma^2 = (\tau\sigma)^q = I$ where I is the identity transformation, and all other relations in σ and τ are consequences of these two. It follows that Γ_q is isomorphic as a group to the free product of cyclic groups $C_2 * C_q$ (see [8]). Of particular importance amongst the Hecke groups is Γ_3 , which is the modular group Γ introduced in Section 1.1. In this sense, the Hecke groups can be considered as generalisations of the modular group; indeed, in some literature the Hecke groups are referred to as the Hecke modular groups.

Consider a finite Rosen continued fraction $[b_1, \dots, b_n]_q$. As in Section 1.2, define a sequence of Möbius transformations

$$f_i(z) = b_i \lambda_q - \frac{1}{z},$$

for $i = 1, 2, \dots, n$, that is, $f_i = \tau^{b_i} \sigma$ using the generators τ and σ of Γ_q . Then the convergents of $[b_1, \dots, b_n]_q$ are given by

$$v_k = f_1 \cdots f_k(\infty),$$

for $k = 1, \dots, n$, and a Rosen continued fraction can be viewed as a sequence $F_k = f_1 \cdots f_k$ of Möbius transformations belonging to Hecke groups. The action of the Hecke groups on \mathbb{H} gives rise to a geometric representation of Rosen continued fractions.

2.1.2 THE FAREY GRAPHS \mathcal{F}_q

The actions of the Hecke groups Γ_q on \mathbb{H} give rise to a class of graphs in which paths represent Rosen continued fractions. Before defining these graphs, we first discuss some geometric properties of the Hecke groups. It is more convenient to work with an alternative pair of generators of Γ_q , namely τ and ρ where

$$\rho(z) = \tau\sigma(z) = \lambda_q - \frac{1}{z}.$$

In \mathbb{H} , the hyperbolic quadrilateral D with vertices i , $e^{i\pi/q}$, $\lambda_q + i$ and the ideal vertex ∞ is a fundamental domain for Γ_q with side-pairing transformations τ and ρ [12, 18]. The quadrilateral D , with $q = 5$, is shown in Figure 2.1 (a).

Let Θ_q denote the group generated by the involutions $\rho^i \sigma \rho^{-i}$, $i = 0, \dots, q-1$, which is a normal subgroup of Γ_q of index q . A fundamental domain E for Θ_q is given by $E = \bigcup_{i=0}^{q-1} \rho^i(D)$, as shown in Figure 2.1 (b) (see [18]). This fundamental domain is an ideal hyperbolic q -gon and its images under Θ_q tessellate the hyperbolic plane by ideal hyperbolic q -gons. The skeleton of this tessellation is a connected plane graph, which we call a *Farey graph*, and denote by \mathcal{F}_q .

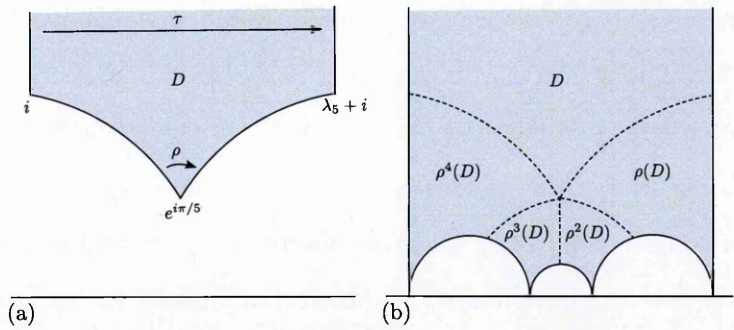


Figure 2.1: (a) A fundamental domain for Γ_5 (b) A fundamental domain for Θ_5

The vertices of \mathcal{F}_q are the ideal vertices of the tessellation; they belong to the ideal boundary $\hat{\mathbb{R}}$ of \mathbb{H} , and in fact form a countable, dense subset of $\hat{\mathbb{R}}$. They are the full set of parabolic fixed points of Γ_q . The edges of \mathcal{F}_q are the sides of the ideal q -gons in the tessellation, and the faces of \mathcal{F}_q are the q -gons themselves. Part of \mathcal{F}_5 is shown in Figure 2.2.

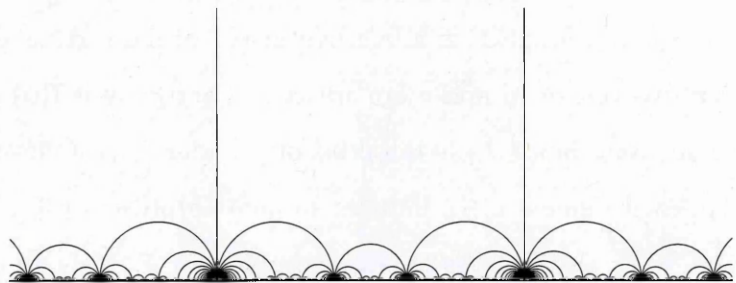


Figure 2.2: The Farey graph \mathcal{F}_5

Notice that the graph \mathcal{F}_3 is the Farey graph \mathcal{F} described in Section 1.2, which has been used in works such as [3] to study integer continued fractions. The Farey graphs \mathcal{F}_q , for all values of q , arise in other subjects involving hyperbolic geometry that are not directly related to continued fractions; for example, they form a class of universal objects in the theory of maps on surfaces (see [26]), and can be used in the study of Diophantine approximation on Hecke groups [18]. The graphs \mathcal{F}_q have not, however, been used previously to study Rosen continued fractions.

There is an alternative way to define \mathcal{F}_q to that given above. Let δ be the hyperbolic geodesic in the upper half-plane \mathbb{H} between 0 and ∞ . Under iterates of the map $\rho(z) = \lambda_q - 1/z$, this hyperbolic line is mapped to each of the q sides of the fundamental domain E of the normal subgroup Θ_q of Γ_q that was described earlier and is shown in Figure 2.1. It follows that \mathcal{F}_q is the orbit of δ under Γ_q , and could have been defined in this way. The transformation $\sigma(z) = -1/z$ maps ∞ to 0, so the set of vertices of \mathcal{F}_q is the orbit of ∞ under Γ_q .

Using this description of \mathcal{F}_q we can determine the neighbours of ∞ in \mathcal{F}_q . Let $\text{Stab}_{\Gamma_q}(\infty)$ be the stabiliser of ∞ in Γ_q ; this is the cyclic group generated by $\tau(z) = z + \lambda_q$. A vertex y is a neighbour of ∞ if and only if $y = g(0)$ for some $g \in \text{Stab}_{\Gamma_q}(\infty)$. Therefore the neighbours of ∞ are the integer multiples of λ_q .

We can also use this alternative description of the Farey graphs to determine the automorphism groups of these graphs. Recall that an *automorphism* of a graph G is a bijective map f of the vertices of G such that two vertices u and v are adjacent if and only if $f(u)$ and $f(v)$ are adjacent. Since \mathcal{F}_q is the orbit of δ under Γ_q , it follows at once that each element of Γ_q induces an automorphism of \mathcal{F}_q . The map $\kappa(z) = -\bar{z}$ also induces an automorphism of \mathcal{F}_q . It is an anti-conformal transformation, so it reverses the cyclic order of vertices around faces, whereas the conformal transformations in Γ_q preserve the cyclic order of vertices around faces. The group generated by Γ_q and κ is the *extended Hecke group*, denoted $\tilde{\Gamma}_q$, and is in fact the full group of automorphisms of \mathcal{F}_q .

Theorem 2.1. *The extended Hecke group $\tilde{\Gamma}_q$ is the group of automorphisms of \mathcal{F}_q .*

Proof. We have already shown that $\tilde{\Gamma}_q$ is a subgroup of the group $\text{Aut}(\mathcal{F}_q)$ of automorphisms of \mathcal{F}_q . We provide only a sketch proof that in fact the two groups are equal. Choose an automorphism f of

\mathcal{F}_q . By post composing f with elements of $\tilde{\Gamma}_q$ we may assume that f fixes ∞ , 0 , and λ_q . A short argument now shows that f must fix each of the vertices of the face of \mathcal{F}_q that is incident to ∞ , 0 , and λ_q , and then an inductive argument shows that f must fix every vertex of \mathcal{F}_q . This shows that f is the identity automorphism, which is induced by the identity element of $\tilde{\Gamma}_q$. \square

The automorphisms of $\tilde{\Gamma}_q$ not only preserve incidence between vertices and edges of \mathcal{F}_q , in fact they also preserve incidence between vertices, edges, and faces. Elements of Γ_q preserve the cyclic order of vertices around faces, whereas elements of $\tilde{\Gamma}_q \setminus \Gamma_q$ reverse this order.

2.1.3 PATHS IN \mathcal{F}_q

We are now in a position to state the theorem that explains the correspondence between Rosen continued fractions and paths in Farey graphs. The case $q = 3$, which is discussed in Section 1.2, has been established already, in [3, Theorem 3.1], and in this case the proof of Theorem 2.2 reduces to the proof of [3, Theorem 3.1]. For now, we will concentrate on finite Rosen continued fractions and finite paths; in Section 2.5 we will consider infinite Rosen continued fractions and paths. Recall from Section 1.1.3 that a path in \mathcal{F}_q is a sequence of vertices v_0, v_1, \dots, v_n , where $v_{i-1} \sim v_i$ for $i = 1, \dots, n$, and that we represent this path by $\langle v_0, v_1, \dots, v_n \rangle$.

Theorem 2.2. *Let y be a vertex of \mathcal{F}_q other than ∞ . Then the vertices v_1, \dots, v_n of \mathcal{F}_q , with $v_n = y$, are the consecutive convergents of some Rosen continued fraction expansion of y if and only if $\langle \infty, v_1, \dots, v_n \rangle$ is a path in \mathcal{F}_q from ∞ to y .*

Proof. Suppose first that v_1, \dots, v_n are the consecutive convergents of the Rosen continued fraction expansion $[b_1, \dots, b_n]_q$ of y . Let

$f_i(z) = b_i\lambda_q - 1/z$, for $i = 1, \dots, n$, so that $v_i = f_1 \cdots f_i(\infty)$. Since $f_1 \cdots f_i$ belongs to the Hecke group Γ_q , $f_1 \cdots f_i(\delta)$ is an edge of \mathcal{F}_q . This edge has one vertex at v_i and the other at

$$f_1 \cdots f_i(0) = f_1 \cdots f_{i-1}(\infty) = v_{i-1}$$

(when $i = 1$ the other vertex is ∞). Therefore v_i and v_{i-1} are adjacent, so $\langle \infty, v_1, \dots, v_n \rangle$ is a path in \mathcal{F}_q from ∞ to y .

Conversely, suppose that $\langle \infty, v_1, \dots, v_n \rangle$ is a path from ∞ to y . We will prove, by induction, that there is a Rosen continued fraction $[b_1, \dots, b_n]_q$ with convergents v_1, \dots, v_n .

Since $v_1 \sim \infty$, there is an integer b_1 such that $v_1 = b_1\lambda_q$, so $v_1 = [b_1]_q$. Suppose now that integers b_1, \dots, b_k have been chosen such that $[b_1, \dots, b_i]_q = v_i$ for $i = 1, \dots, k$, where $k < n$. As usual, let $f_i(z) = b_i\lambda_q - 1/z$. Since $v_k \sim v_{k+1}$, and since the maps f_i are automorphisms of \mathcal{F}_q , we see that $f_k^{-1} \cdots f_1^{-1}(v_k) \sim f_k^{-1} \cdots f_1^{-1}(v_{k+1})$. Now, $f_k^{-1} \cdots f_1^{-1}(v_k)$ is equal to ∞ , which implies that $f_k^{-1} \cdots f_1^{-1}(v_{k+1})$ is given by $b_{k+1}\lambda_q$, for some integer b_{k+1} . We then define $f_{k+1}(z) = b_{k+1}\lambda_q - 1/z$, so that

$$v_{k+1} = f_1 \cdots f_k(b_{k+1}\lambda_q) = f_1 \cdots f_k f_{k+1}(\infty).$$

Therefore, by induction, v_1, \dots, v_n are the consecutive convergents of a Rosen continued fraction expansion of y . \square

As in Section 1.2, we call $\langle \infty, v_1, \dots, v_n \rangle$ the *path of convergents* of $[b_1, \dots, b_n]_q$. There is a simple way to move between a Rosen continued fraction and its path of convergents, which we outline here, and which is illustrated in Figure 2.3. The integers b_2, \dots, b_n of the expansion $[b_1, \dots, b_n]_q$ of a vertex y encode a set of directions to navigate the corresponding path $\langle \infty, v_1, \dots, v_n \rangle$. In brief, to navigate the path

you should upon reaching v_{i-1} take the ' b_i th right turn' to get to v_i (which is a left turn if b_i is negative).

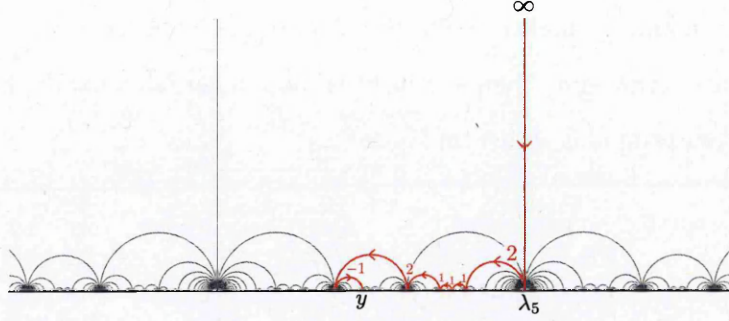


Figure 2.3: The route of the path tells us that $y = [1, 2, 1, 1, 1, 2, -1]_5$

Let us describe this procedure in more detail. Suppose that a , b , and c are vertices of \mathcal{F}_q such that $a \sim b$ and $b \sim c$. We are going to define an integer valued function $\phi(a, b, c)$. Suppose first that $b = \infty$; then $\phi(a, \infty, c) = (c - a)/\lambda_q$. Now suppose that $b \neq \infty$. In this case we choose an element f of Γ_q such that $f(b) = \infty$ and define $\phi(a, b, c) = \phi(f(a), f(b), f(c))$. The choice of f does not matter, because if g is another element of Γ_q such that $g(b) = \infty$, then $g = \tau^m f$ for some integer m , and hence

$$\begin{aligned} \phi(g(a), g(b), g(c)) &= (g(c) - g(a))/\lambda_q \\ &= ((m + f(c)) - (m + f(a)))/\lambda_q \\ &= (f(c) - f(a))/\lambda_q \\ &= \phi(f(a), f(b), f(c)). \end{aligned}$$

A consequence of this definition is that ϕ is invariant under elements of Γ_q , in the sense that

$$\phi(f(a), f(b), f(c)) = \phi(a, b, c)$$

for any map f in Γ_q and three vertices a , b , and c .

The function ϕ has a simple geometric interpretation when $b \neq \infty$, which we obtain using the conformal action of the automorphisms Γ_q . Label the edges incident to b by the integers, with $[a, b]$ (the edge between a and b) labelled 0, and the other edges labelled in anticlockwise order around b . Then $\phi(a, b, c)$ is the integer label for the edge $[b, c]$. An example is shown in Figure 2.4

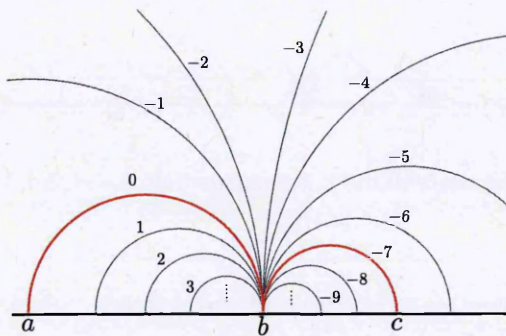


Figure 2.4: An example in which $\phi(a, b, c) = -7$

We finish here with a lemma that explains precisely how the integers b_1, \dots, b_n encode a set of directions for navigating a path in \mathcal{F}_q .

Lemma 2.3. *Let $\langle v_0, v_1, \dots, v_n \rangle$, where $v_0 = \infty$ and $n \geq 2$, be the path of convergents of the Rosen continued fraction $[b_1, \dots, b_n]_q$. Then $\phi(v_{i-2}, v_{i-1}, v_i) = b_i$ for $i = 2, \dots, n$.*

Proof. Since ϕ is invariant under the element $f_1 \cdots f_{i-1}$ of Γ_q , we see that

$$\begin{aligned} \phi(v_{i-2}, v_{i-1}, v_i) &= \phi(f_1 \cdots f_{i-2}(\infty), f_1 \cdots f_{i-1}(\infty), f_1 \cdots f_i(\infty)) \\ &= \phi(f_{i-1}^{-1}(\infty), \infty, f_i(\infty)) \\ &= \phi(0, \infty, b_i \lambda_q) \\ &= b_i. \end{aligned}$$

□

2.2 GEODESIC ROSEN CONTINUED FRACTIONS

Each vertex y of \mathcal{F}_q has infinitely many Rosen continued fraction expansions. In the familiar case $q = 3$, in which the Rosen continued fractions have integer coefficients, there are numerous algorithms that give rise to different continued fraction expansions of the rational y . The most well known of these is Euclid's algorithm, which gives the regular continued fraction expansion of y . A similar algorithm is the nearest-integer algorithm. It has been known for a long time (see [39, page 168]) that among all integer continued fraction expansions of a rational number y , the one arising from the nearest-integer algorithm has the least number of terms.

Rosen observed in [41] that given a vertex y of \mathcal{F}_q , there is a version of the nearest-integer algorithm that gives rise to a finite Rosen continued fraction expansion of y . It is much the same as the more familiar nearest-integer algorithm that is used with integer continued fractions, and similar algorithms can be found in, for example, [36]. We say that a finite Rosen continued fraction expansion $[b_1, \dots, b_n]_q$ of a vertex y is *geodesic* if every other Rosen continued fraction expansion of y has at least n terms. In this section we show that applying the nearest-integer algorithm to a vertex y of \mathcal{F}_q produces a geodesic Rosen continued fraction expansion of y .

In this section we will discuss only finite Rosen continued fractions. The nearest-integer algorithm only yields a finite continued fraction if it is applied to a number y that is a vertex of \mathcal{F}_q . Precisely which real numbers are vertices of \mathcal{F}_q (or, equivalently, are parabolic fixed points of Γ_q) has been studied in a number of works, including [38], [47] and [63]. It is also the case that when y is not a vertex of \mathcal{F}_q , the nearest-integer algorithm applied to y gives rise to an infinite

geodesic Rosen continued fraction; exactly what this means and why it is true will be discussed in Section 2.5.

2.2.1 GEODESIC PATHS IN \mathcal{F}

A geodesic Rosen continued fraction is so called because its corresponding path in \mathcal{F}_q is geodesic, that is, a path with the least number of edges. We begin by describing an algorithm for constructing a geodesic path between two vertices of a Farey graph \mathcal{F}_q . When $q = 3$, our algorithm coincides with that of [3, Section 9]. We will then show, in Section 2.2.2, that when one of the vertices is ∞ , the algorithm reduces to the nearest-integer algorithm, thereby proving that if y is a vertex of \mathcal{F}_q , the nearest-integer algorithm gives rise to geodesic Rosen continued fraction expansion of y .

As we have seen, the edges of the Farey graph \mathcal{F}_q lie in the upper half-plane \mathbb{H} and the vertices of \mathcal{F}_q lie on the ideal boundary of \mathbb{H} , $\hat{\mathbb{R}}$. We can map \mathbb{H} conformally on to the unit disc in \mathbb{C} by a Möbius transformation, and under such a transformation the ideal boundary $\hat{\mathbb{R}}$ maps to the unit circle. It is often more convenient to think of \mathcal{F}_q as a graph in the unit disc, not least because in that model it is obvious geometrically that the ideal boundary is topologically a circle, and we can speak of a finite list of points on the circle occurring in 'clockwise order'.

The Möbius transformation we use to transfer \mathcal{F}_q to the unit disc is

$$\psi(z) = \frac{z - e^{i\pi/q}}{z - e^{-i\pi/q}},$$

because this maps $e^{i\pi/q}$, one of the centres of rotation of the generator ρ of Γ_q , to 0. The Farey graph \mathcal{F}_5 is shown in the unit disc in Figure 2.5.

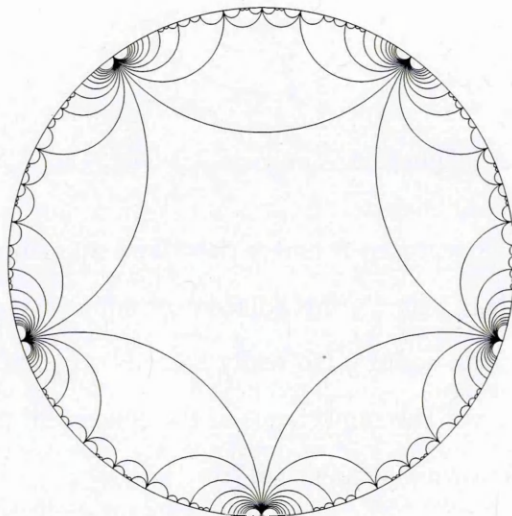


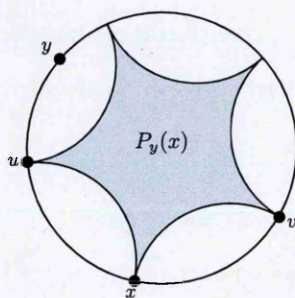
Figure 2.5: The Farey graph \mathcal{F}_5

Viewing \mathcal{F}_q in the unit disc allows us to more easily describe an algorithm for constructing a geodesic path in \mathcal{F}_q . At the heart of this algorithm is the following lemma.

Lemma 2.4. *Let x and y be two distinct, non-adjacent vertices of \mathcal{F}_q . Among the faces of \mathcal{F}_q that are incident to x , there is a unique one P such that if u and v are the two vertices of P that are adjacent to x , then y belongs to the component of $\hat{\mathbb{R}} \setminus \{u, v\}$ that does not contain x .*

Proof. We prove the lemma in \mathbb{H} where, after applying an element of Γ_q , we may assume that $x = \infty$ and $y \in (0, \lambda_q)$. With this choice of x and y , the unique polygon P is the fundamental domain E of the normal subgroup Θ_q of Γ_q defined in Section 2.1, which is shown in Figure 2.1. □

We denote the polygon P described in Lemma 2.4 by $P_y(x)$; an example is shown in Figure 2.6.

Figure 2.6: A polygon $P_y(x)$ in \mathcal{F}_5

We call the two vertices u and v described in [Lemma 2.4](#) the y -parents of x . When x and y are adjacent or equal, we define the y -parents of x to both equal y (so really there is only one y -parent in each of these cases). The importance of the concept of y -parents can be seen in the following sequence of four results.

Lemma 2.5. *Let u and v be distinct vertices of a face P of \mathcal{F}_q . Suppose that γ is a path in \mathcal{F}_q that starts at a vertex in one component of $\hat{\mathbb{R}} \setminus \{u, v\}$ and finishes at a vertex in the other component of $\hat{\mathbb{R}} \setminus \{u, v\}$. Then γ passes through one of u or v .*

Proof. If the lemma is false, then there is an edge of γ with endpoints in each of the components of $\hat{\mathbb{R}} \setminus \{u, v\}$. This edge intersects the hyperbolic line between u and v , which is a contradiction, because this hyperbolic line either lies inside the face P or else it is an edge of P . □

Theorem 2.6. *Any path between vertices x and y of \mathcal{F}_q must pass through one of the y -parents of x .*

Proof. This is certainly true if x and y are adjacent, as in this case the y -parents of x both equal y . Otherwise, the theorem follows immediately from [Lemma 2.4](#) and [Lemma 2.5](#). □

Corollary 2.7. *If $\langle v_0, \dots, v_n \rangle$ is a geodesic path in \mathcal{F}_q with $v_0 = x$ and $v_n = y$, where $n \geq 1$, then v_1 is a y -parent of x .*

Proof. By Theorem 2.6, we can choose a smallest integer $i \geq 1$ such that v_i is a y -parent of x . Then i must be 1, otherwise the path is not a geodesic path because $\langle v_0, v_i, v_{i+1}, \dots, v_n \rangle$ is a shorter path between x and y . \square

Corollary 2.7 itself has a corollary that we will need soon.

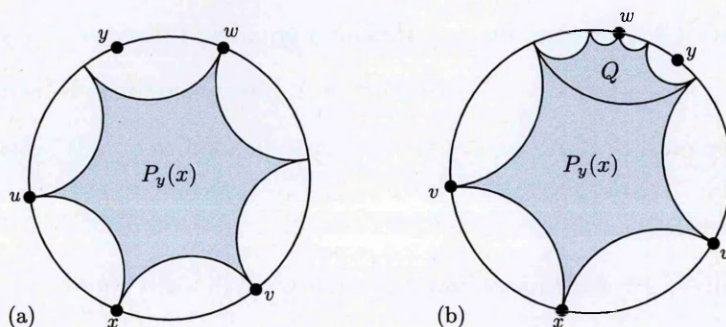
Corollary 2.8. *Each geodesic path between vertices x and y of a face P of \mathcal{F}_q is given by traversing the edges of P . There is a unique geodesic path between x and y unless q is even and x and y are opposite vertices of P , in which case there are exactly two geodesic paths between x and y .*

Proof. This is clearly true if x and y are adjacent. Otherwise, notice that the y -parents of x both lie in P . It then follows from Corollary 2.7 that any geodesic path between x and y is confined to the vertices of P , and the result follows immediately. \square

We denote the y -parents of x by $\alpha_y(x)$ and $\beta_y(x)$ in an order that we now explain. If x and y are equal or adjacent, then we must define $\alpha_y(x)$ and $\beta_y(x)$ to both equal y . For the remaining possibilities, we split our discussion into two cases, depending on whether q is even or odd.

Suppose first that q is even. In this case there is a vertex w of the q -gon $P_y(x)$ defined after Lemma 2.4 that is opposite x . If $y = w$, then we define $\alpha_y(x)$ and $\beta_y(x)$ to be such that the vertices $\alpha_y(x), x, \beta_y(x)$ lie in that order clockwise around \mathbb{R} . If $y \neq w$, then we define $\alpha_y(x)$ to be whichever of the y -parents u and v of x lies in the same component of $\mathbb{R} \setminus \{x, w\}$ as y , as shown in Figure 2.7. Of course, $\beta_y(x)$ is then the remaining vertex u or v .

Suppose now that q is odd. In this case, there is an edge of the q -gon $P_y(x)$ that is opposite x , rather than a vertex. Let Q be the other face of \mathcal{F}_q that is incident to that edge. Together $P_y(x)$ and Q form a $(2q - 2)$ -gon, and we now define $\alpha_y(x)$ and $\beta_y(x)$ using this

Figure 2.7: The vertex u is $\alpha_y(x)$

$(2q - 2)$ -gon in the same way that we did when q was even, as shown in Figure 2.7.

We have now defined two maps $\alpha_y : \mathcal{F}_q \rightarrow \mathcal{F}_q$ and $\beta_y : \mathcal{F}_q \rightarrow \mathcal{F}_q$. By definition, they are invariant under Γ_q in the sense that $f(\alpha_y(x)) = \alpha_{f(y)}(f(x))$ and $f(\beta_y(x)) = \beta_{f(y)}(f(x))$ for any transformation f from Γ_q . When $q = 3$ and $y = \infty$, the vertices $\alpha_y(x)$ and $\beta_y(x)$ have been given various names in continued fractions literature. In [3] they were called the *first parent* and *second parent* of x and in [30] they were called the *old parent* and *young parent* of x .

From Corollary 2.7 we can see that every geodesic path from x to y is given by applying some sequence of the maps α_y and β_y successively to the vertex x . In fact, we will now show that applying the map α_y repeatedly gives a geodesic path from x to y . We first define the *graph metric* d_q on \mathcal{F}_q as follows. Given vertices x and y of \mathcal{F}_q , $d_q(x, y)$ is the length of a geodesic path between x and y .

Lemma 2.9. *Suppose that x and y are distinct vertices of \mathcal{F}_q . Then*

$$d_q(\alpha_y(x), y) = d_q(x, y) - 1.$$

Proof. Let $n = d_q(x, y)$. The result is immediate if $n = 1$, so let us assume that $n \geq 2$. Let $\gamma = \langle v_0, \dots, v_n \rangle$ be any geodesic path with $v_0 = x$ and $v_n = y$. By Corollary 2.7 we know that v_1 is equal to either $\alpha_y(x)$ or $\beta_y(x)$. If the former is true, then our result is proved, so let us

suppose instead that $v_1 = \beta_y(x)$, in which case $d_q(\beta_y(x), y) = n - 1$. We split our argument into two cases depending on whether q is even or odd.

If q is even then there is a vertex w opposite x on the face $P_y(x)$. The vertex y lies in the opposite component of $\hat{\mathbb{R}} \setminus \{x, w\}$ to $\beta_y(x)$ (or possibly $y = w$) so Lemma 2.5 implies that γ must pass through either x or w . Since, by Corollary 2.8, x and w are equally close to $\alpha_y(x)$ as they are to $\beta_y(x)$ we see that

$$d_q(\alpha_y(x), y) = d_q(\beta_y(x), y) = n - 1.$$

Suppose now that q is odd. Let e be the edge of $P_y(x)$ opposite x , and let Q be the other face of \mathcal{F}_q incident to e . Let w be the vertex opposite x in the $(2q - 2)$ -gon formed by joining $P_y(x)$ and Q , as shown in Figure 2.7 (b). We also define a and b to be the vertices incident to e , where a lies in the same component of $\hat{\mathbb{R}} \setminus \{x, w\}$ as $\alpha_y(x)$. The vertex y either lies in the opposite component of $\hat{\mathbb{R}} \setminus \{x, a\}$ to $\beta_y(x)$ or else it lies in the opposite component of $\hat{\mathbb{R}} \setminus \{a, w\}$ to $\beta_y(x)$ (or possibly y equals a or w). Lemma 2.5 implies that γ must pass through either x , a , or w . Since, by Corollary 2.8, x , a , and w are at least as close to $\alpha_y(x)$ as they are to $\beta_y(x)$ we see that, once again,

$$d_q(\alpha_y(x), y) = d_q(\beta_y(x), y) = n - 1.$$

□

The next corollary is an immediate consequence of this lemma.

Corollary 2.10. *Suppose that x and y are distinct vertices of \mathcal{F}_q . Then there is a positive integer m such that $\alpha_y^m(x) = y$, and $\langle x, \alpha_y(x), \alpha_y^2(x), \dots, \alpha_y^m(x) \rangle$ is a geodesic path from x to y .*

2.2.2 THE NEAREST-INTEGERS CONTINUED FRACTION ALGORITHM

Corollary 2.10 tells us that to construct a geodesic path between two vertices x and y of \mathcal{F}_q , we repeatedly apply the map α_y . Next we shall show that when $x = \infty$, this geodesic path is the same as the path that arises from applying the nearest-integer algorithm. First we must describe that algorithm. Our description is essentially equivalent to Rosen's [41], but couched in the language of this thesis.

We will show how to apply the nearest-integer algorithm to a real number y to give a Rosen continued fraction in an inductive manner. For now we assume that y is a vertex of \mathcal{F}_q , so that the continued fraction is finite. In Section 2.5 we will discuss the same algorithm when y is not a vertex of \mathcal{F}_q . We will use the notation $\|x\|_q$ to denote the nearest-integer multiple of λ_q to x , and we will make the convention that for an integer m , $\|(m + 1/2)\lambda_q\|_q = m\lambda_q$.

Let $b_1\lambda_q = \|y\|_q$, where $b_1 \in \mathbb{Z}$. We define $f_1(z) = b_1\lambda_q - 1/z$. Suppose that we have constructed a sequence of integers b_1, \dots, b_k and a corresponding sequence of maps $f_i(z) = b_i\lambda_q - 1/z$, for $i = 1, \dots, k$. Then we define $b_{k+1}\lambda_q = \|f_k^{-1} \cdots f_1^{-1}(y)\|_q$, where $b_{k+1} \in \mathbb{Z}$, provided $f_k^{-1} \cdots f_1^{-1}(y)$ is not ∞ . We then define $f_{k+1}(z) = b_{k+1}\lambda_q - 1/z$.

We will prove shortly that $f_m^{-1} \cdots f_1^{-1}(y) = \infty$ for some positive integer m , and at this stage the algorithm terminates. The outcome is a sequence of integers b_1, \dots, b_m and maps f_1, \dots, f_m . Since $y = f_1 \cdots f_m(\infty)$, we see that the continued fraction $[b_1, \dots, b_m]_q$ has value y ; it is called the *nearest-integer continued fraction expansion* of y .

Let us now see why the nearest-integer algorithm applied to a vertex y is equivalent to iterating the map α_y . The key to this equivalence is the following lemma.

Lemma 2.11. *Suppose that y is a vertex of \mathcal{F}_q other than ∞ . Then $\alpha_y(\infty) = b\lambda_q$, for some integer b , where $b\lambda_q = \|y\|_q$.*

Proof. The result is immediate if y is an integer multiple of λ_q – a neighbour of ∞ – so let us assume that this is not so. Then y lies between $b\lambda_q$ and $(b+1)\lambda_q$, for some integer b . Therefore $P_y(\infty)$ is the face of \mathcal{F}_q that is incident to ∞ , $b\lambda_q$, and $(b+1)\lambda_q$, which implies that $\alpha_y(\infty)$ is equal to either $b\lambda_q$ or $(b+1)\lambda_q$. Using the generators τ and ρ we can calculate all the vertices of $P_y(\infty)$ explicitly. Other than ∞ , they are given by

$$b\lambda_q + \frac{\sin(\pi(j-1)/q)}{\sin(\pi j/q)}, \quad \text{for } j = 1, \dots, q-1.$$

When q is even, the vertex opposite ∞ in $P_y(\infty)$ is $w = (b+1/2)\lambda_q$. If $y = w$, then $\alpha_y(\infty) = b\lambda_q$, because $b\lambda_q, \infty, (b+1)\lambda_q$ lie in that order clockwise around $\hat{\mathbb{R}}$. So in this case $\alpha_y(\infty) = b\lambda_q$. If $y \neq w$, then $\alpha_y(\infty)$ lies in the same component of $\mathbb{R} \setminus \{w\}$ as y , so again it is equal to $b\lambda_q$.

When q is odd, the argument is similar, if slightly more involved: we construct the face Q as we did in Figure 2.7, and determine that the vertex opposite ∞ in the $(2q-2)$ -gon made up of $P_y(\infty)$ joined to Q is again $w = (b+1/2)\lambda_q$. We then proceed as before; the details are omitted. \square

Theorem 2.12. *Let b_1, b_2, \dots be the sequence of integers and f_1, f_2, \dots the sequence of maps that arise in applying the nearest-integer algorithm to a real number y that is a vertex of \mathcal{F}_q . Suppose that $d_q(\infty, y) = m$. Then $f_1 \cdots f_k(\infty) = \alpha_y^k(\infty)$ for $k = 1, \dots, m$. In particular, the nearest-integer algorithm terminates.*

Proof. We proceed by induction on k . First,

$$f_1(\infty) = b_1\lambda_q = \alpha_y(\infty),$$

using Lemma 2.11 for the second equality. Now suppose that $f_1 \cdots f_k(\infty) = \alpha_y^k(\infty)$, where $k < m$. Then

$$\alpha_y^{k+1}(\infty) = \alpha_y(f_1 \cdots f_k(\infty)) = f_1 \cdots f_k(\alpha_{f_k^{-1} \cdots f_1^{-1}(y)}(\infty)),$$

using invariance of α under the transformation $f_1 \cdots f_k$ (an element of Γ_q) for the second equality. The vertex $f_k^{-1} \cdots f_1^{-1}(y)$ of \mathcal{F}_q is not ∞ , for if it were then $y = f_1 \cdots f_k(\infty) = \alpha_y^k(\infty)$. By definition, then, $b_{k+1}\lambda_q$ is the nearest integer multiple of λ_q to $f_k^{-1} \cdots f_1^{-1}(y)$. Therefore, using Lemma 2.11 again,

$$\alpha_{f_k^{-1} \cdots f_1^{-1}(y)}(\infty) = b_{k+1}\lambda_q = f_{k+1}(\infty).$$

Hence

$$\alpha_y^{k+1}(\infty) = f_1 \cdots f_{k+1}(\infty),$$

which completes the inductive step. \square

We can now prove that the nearest-integer algorithm gives rise to a geodesic Rosen continued fraction expansion of a vertex of \mathcal{F}_q .

Theorem 2.13. *For each $q \geq 3$, the nearest-integer algorithm applied to a vertex y of \mathcal{F}_q gives rise to a geodesic Rosen continued fraction expansion of y .*

Proof. Let y be a vertex of \mathcal{F}_q . Corollary 2.10 tells us that there is a positive integer m such that $\alpha_y^m(\infty) = y$ and

$$\langle \infty, \alpha_y(\infty), \alpha_y^2(\infty), \dots, \alpha_y^m(\infty) \rangle$$

is a geodesic path. Theorem 2.12 says that this path is the same path as the path of convergents from ∞ to y given by the nearest-integer con-

tinued fraction expansion of y . Therefore this expansion is a geodesic Rosen continued fraction expansion. \square

2.2.3 EQUIVALENT PATHS

In [41], Rosen described a sequence of operations that can be used to transform any Rosen continued fraction with value y to the nearest-integer expansion of y . We end this section by explaining informally how this process can be illuminated using paths in \mathcal{F}_q , without engaging with the details of Rosen's arguments. We will show that any path from ∞ to y is "homotopic" (in a sense that will shortly be made precise) to the path of convergents from ∞ to y that arises from the nearest-integer algorithm. We will use only elementary graph theory.

To explain our method, let us begin with a finite connected plane graph X and a path γ in X . We define two elementary operations that can be applied to γ . The first is to either insert or remove a subpath of length two that proceeds from one vertex to a neighbouring vertex and immediately back again. The second is to either insert or remove a subpath that consists of a full circuit of the boundary of a face. These two operations preserve the start and endpoints of γ . They are illustrated in Figure 2.8.

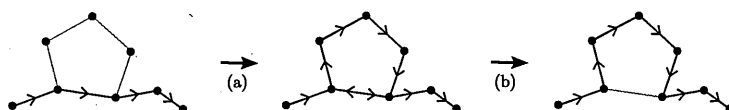


Figure 2.8: (a) A clockwise circuit of the boundary of a face is inserted into the path (b) A subpath that proceeds from one vertex to a neighbouring vertex and immediately back again is removed from the path

If we can transform γ to another path γ' by a finite sequence of these elementary operations, then we say that γ and γ' are *homotopic*. There is a theory of homotopic paths for more general 2-complexes

than just finite connected plane graphs; see, for example, [9, Section 1.2]. Using this concept of homotopy, one can define the fundamental group of a 2-complex in the obvious way, and, as you might expect, the fundamental group of a finite connected plane graph is trivial (see [9, Corollary 1.2.14]). It follows that any two paths in a finite connected plane graph that start at the same vertex and finish at the same vertex are homotopic.

We can define homotopy for finite paths in the infinite graph \mathcal{F}_q in exactly the same way as we have done for finite connected plane graphs, and we obtain the same conclusion about homotopic paths.

Theorem 2.14. *Any two paths in \mathcal{F}_q that start at the same vertex and finish at the same vertex are homotopic.*

Proof. Suppose that we start with a single face of \mathcal{F}_q , then adjoin all neighbouring faces, then adjoin all neighbouring faces of all those faces, and so forth. This yields a sequence of finite connected plane subgraphs of \mathcal{F}_q . We can choose a subgraph sufficiently far along the sequence that it contains all the vertices of the two paths. The two paths are homotopic in this subgraph, so they are homotopic in \mathcal{F}_q . \square

Theorem 2.14 shows that any two paths from ∞ to a vertex y of \mathcal{F}_q are homotopic, so in particular, any path from ∞ to y is homotopic to the path that we obtain by applying the nearest-integer algorithm to y . Using Lemma 2.3 we can reinterpret the two elementary operations as transformations of the Rosen continued fractions corresponding to the paths. This is illustrated by Figure 2.9. The first elementary operation corresponds to inserting or removing a 0 coefficient in the continued fraction, and the second operation corresponds to inserting or removing $q - 1$ consecutive coefficients of value 1 or $q - 1$ consecutive coefficients of value -1 from the continued fraction.

(Both operations also impact on the neighbouring coefficients in the continued fraction; we will not go into this.) These two operations are essentially the same operations that Rosen uses in [41]. They can be seen as applications of the two relations $\sigma^2 = I$ and $\rho^q = I$ satisfied by the generators σ and ρ of Γ_q .

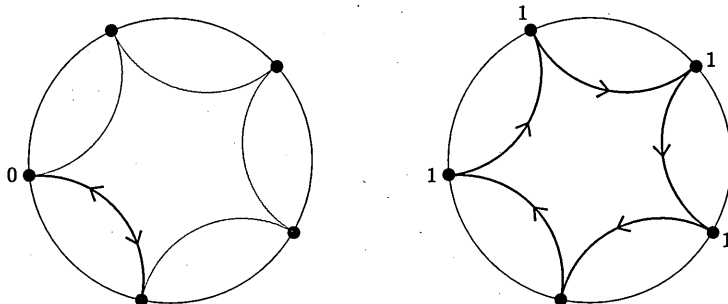


Figure 2.9: Two paths in \mathcal{F}_q labelled with coefficients of the corresponding continued fractions

2.3 THE NUMBER OF GEODESIC ROSEN CONTINUED FRACTIONS

It follows immediately from Corollary 2.8 that a geodesic Rosen continued fraction expansion of a vertex y of \mathcal{F}_q is not necessarily unique. The question arises as to precisely how many geodesic expansions of y there may be. We observe that for each vertex y of \mathcal{F}_q , we can “shade in” each of the faces of \mathcal{F}_q that separates ∞ from y , as shown in Figure 2.10.

This results in a chain of q -gons, with ∞ a vertex of the first q -gon and y a vertex of the last q -gon. It is often clearer to represent the chain of q -gons by polygons of a similar Euclidean size, such as those shown in Figure 2.11 (the first four q -gons of this chain match those of Figure 2.10).

We define $D_q(\infty, y)$ to be the number of q -gons in this chain. We will show that every geodesic path between ∞ and y consists of edges

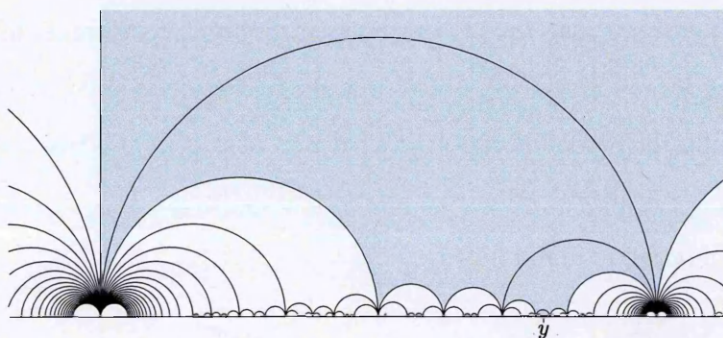


Figure 2.10: The ideal q -gons that separate ∞ and y are shaded

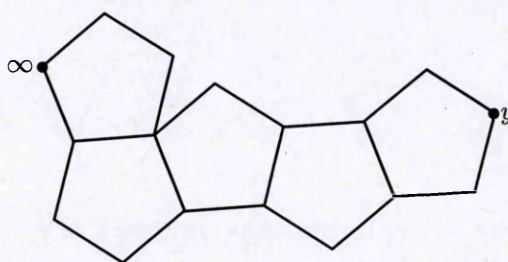


Figure 2.11: A chain of 5-gons connecting ∞ and y

belonging to faces in the chain of q -gons described above. This restriction allows us to prove sharp upper bounds on the number of geodesic Rosen continued fraction expansions of y in terms of both $D_q(\infty, y)$ and in terms of $d_q(\infty, y)$, the length of a geodesic path between x and y .

2.3.1 q -CHAINS

Let us now properly define the chain of q -gons between two non-adjacent vertices x and y of \mathcal{F}_q that was introduced informally above. Consider a collection of Euclidean q -gons P_1, \dots, P_n in the plane such that P_{i-1} and P_i have a common edge for $i = 2, \dots, n$ but otherwise the q -gons (including their interiors) do not overlap one another. Together these q -gons give rise to a connected, finite plane graph called a q -chain whose vertices and edges are those of the constituent q -gons. For example, a 5-chain is shown in Figure 2.11. We

also refer to plane graphs that are topologically equivalent to q -chains as q -chains.

Next we describe a process for constructing a q -chain P_1, \dots, P_n consisting of faces of \mathcal{F}_q such that x is a vertex of P_1 and y is a vertex of P_n . First, let P_1 be the face $P_y(x)$ (which was defined after [Lemma 2.4](#)). If y is a vertex of P_1 , then the construction terminates. Otherwise, there are two adjacent vertices a_1 and b_1 of P_1 such that y belongs to the component of $\mathbb{R} \setminus \{a_1, b_1\}$ that contains no other vertices of P_1 . Define P_2 to be the face of \mathcal{F}_q other than P_1 that is also incident to the edge $\{a_1, b_1\}$. If y is a vertex of P_2 , then the construction terminates. Otherwise, there are two adjacent vertices a_2 and b_2 of P_2 such that y belongs to the component of $\mathbb{R} \setminus \{a_2, b_2\}$ that contains no other vertices of P_2 . We then define P_3 to be the face of \mathcal{F}_q other than P_2 that is incident to $\{a_2, b_2\}$, and the procedure continues in this fashion. The resulting sequence of q -gons is uniquely defined by this process, because there is only one choice for each pair $\{a_i, b_i\}$. The first few q -gons in such a sequence are shown in [Figure 2.12](#).

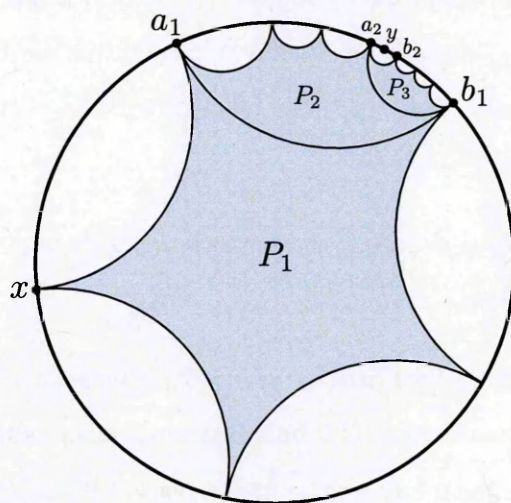


Figure 2.12: The first three q -gons in a q -chain

We must show that the procedure terminates. To this end, let us first show that a vertex v of \mathcal{F}_q can be incident to only finitely many

consecutive faces of P_1, P_2, \dots . By applying a suitable element of Γ_q , we see that it suffices to prove this when $v = \infty$. The faces of \mathcal{F}_q incident to ∞ are just the translates by iterates of τ of the fundamental domain E of Θ_q , defined in Section 2.1, and it is straightforward to check that only finitely many of these q -gons can appear consecutively in the sequence P_1, P_2, \dots .

We deduce that there is a sequence of positive integers $n_1 < n_2 < \dots$ such that each pair of edges $\{a_{n_i}, b_{n_i}\}$ and $\{a_{n_{i+1}}, b_{n_{i+1}}\}$ do not have a common vertex. Let d_i be the distance in the graph metric from y to a_{n_i} or b_{n_i} , whichever is nearest. It can easily be checked that x and y lie in distinct components of $\mathbb{R} \setminus \{a_j, b_j\}$, for each edge $\{a_j, b_j\}$, so Lemma 2.5 tells us that any path from x to y must pass through one of a_j or b_j . It follows that d_1, d_2, \dots is a decreasing sequence of positive integers, which must eventually terminate. Therefore the sequence P_1, P_2, \dots has a final member P_n , which is incident to y .

The resulting sequence P_1, \dots, P_n is a q -chain that is a subgraph of \mathcal{F}_q , which we call the q -chain from x to y . Figure 2.12 shows that using the disc model of the hyperbolic plane, only a few faces from a q -chain are large enough (in Euclidean terms) that we can see them. Instead we usually draw q -chains using Euclidean polygons, as we did in Figure 2.11.

Lemma 2.15. *The y -parents of any vertex in the q -chain from x to y also belong to the q -chain.*

Proof. Let us denote the chain by P_1, \dots, P_n . Choose a vertex z of this q -chain other than y , and let k be the largest integer such that z is a vertex of P_k . Define u and v to be the vertices of P_k adjacent to z . The point y cannot lie in the component of $\mathbb{R} \setminus \{u, v\}$ that contains z , for if it did then P_{k+1} would contain one of the edges $\{z, u\}$ or $\{z, v\}$, in which case z would be a vertex of P_{k+1} . Therefore either y is equal to u or v , so that y is the single y -parent of z , or otherwise y belongs to

the component of $\hat{\mathbb{R}} \setminus \{u, v\}$ that does not contain z . Then, by definition (see Lemma 2.4), u and v are the parents of z , so in particular they belong to the q -chain. \square

There is an important consequence of this lemma.

Theorem 2.16. *Any geodesic path from a vertex x to another vertex y in \mathcal{F}_q is contained in the q -chain from x to y .*

Proof. This is an immediate consequence of Corollary 2.7 and Lemma 2.15. \square

Theorem 2.16 tells us that to understand geodesic paths in Farey graphs, it suffices to understand geodesic paths in q -chains. This is a significant reduction because q -chains are simple, finite plane graphs.

Let us define a binary function D_q on the vertices of \mathcal{F}_q as follows. If x and y are equal, then $D_q(x, y) = 0$, and if they are adjacent then $D_q(x, y) = 1$. Otherwise, $D_q(x, y)$ is the number of q -gons in the q -chain from x to y . The function D_q is closely related to the graph metric on the dual graph of \mathcal{F}_q . However, although D_q is symmetric, it does not satisfy the triangle inequality (we omit proofs of these two facts as we do not need them).

2.3.2 NUMBER OF GEODESIC ROSEN CONTINUED FRACTIONS IN TERMS OF $D_q(\infty, x)$

We begin by giving an upper bound on the number of geodesic Rosen continued fraction expansions of a vertex y of \mathcal{F}_q in terms of the number of q -gons in the q -chain from ∞ to y . We do so by proving a slightly more general result that gives an upper bound on the number of geodesic paths in \mathcal{F}_q between two vertices x and y with $D_q(x, y) = n$.

We use the following notation. Let P_1, \dots, P_n be the q -chain between non-adjacent vertices x and y of \mathcal{F}_q , and let $\{a_i, b_i\}$ be the edge between P_i and P_{i+1} for $i = 1, \dots, n-1$. We choose a_i and b_i such that a_i, a_{i+1}, b_{i+1} , and b_i occur in that order clockwise around P_{i+1} . It is convenient to also define $a_0 = b_0 = x$ and $a_n = b_n = y$. We define μ_i to be the path on P_i that travels clockwise from a_{i-1} to a_i , and we define ν_i to be the path on P_i that travels anticlockwise from b_{i-1} to b_i . All this notation is illustrated in Figure 2.13, in which the edge between a_i and b_i is labelled by its length 1.

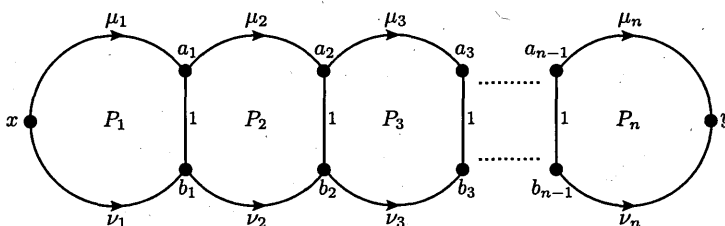


Figure 2.13: A q -chain from x to y

We denote the length of a path γ in \mathcal{F}_q by $|\gamma|$. Note that $|\gamma|$ is the number of edges of γ , not the number of vertices, which is $|\gamma| + 1$.

We define F_i to be the i th term of the Fibonacci sequence, which is given by $F_{-1} = 1, F_0 = 1, F_1 = 2, F_2 = 3, F_3 = 5$, and so forth; note the unusual choice of indices. Let $N(x, y)$ denote the number of geodesic paths from x to y . The following theorem gives bounds on N when x and y are vertices of \mathcal{F}_q with q even; we treat the case in which q is odd separately in Theorem 2.18 and Theorem 2.19.

Theorem 2.17. *Suppose that x and y are vertices of \mathcal{F}_q , for q even, with $D_q(x, y) = n$. Then $N(x, y) \leq F_n$. Furthermore, there are vertices x and y with $D_q(x, y) = n$ for which this bound can be attained.*

Proof. We prove the inequality $N(x, y) \leq F_n$ by using induction on n . It is immediate if n is 0 or 1. Suppose now that $n > 1$, and assume that the inequality is true for all pairs of vertices u and v with $D_q(u, v) <$

n . Choose two vertices x and y with $D_q(x, y) = n$, and let P_1, \dots, P_n be the q -chain from x to y , illustrated in Figure 2.13. Without loss of generality, we may assume that $|\mu_1| \leq |\nu_1|$. If $|\mu_1| < |\nu_1| - 1$, then every geodesic path from x to y must pass along the path μ_1 . Since $D_q(a_1, y) = n - 1$ we see by induction that

$$N(x, y) = N(a_1, y) \leq F_{n-1} < F_n.$$

The remaining possibility is that $|\nu_1| - 1 \leq |\mu_1| \leq |\nu_1|$. In this case, the set of geodesic paths from x to y can be partitioned into the set A of those geodesic paths that travel along the path μ_1 and the set B of those geodesic paths that travel along the path ν_1 . A path in B cannot pass through a_1 as well as b_1 because it is a geodesic path and $|\mu_1| < |\nu_1| + 1$. Instead it must pass through b_2 . Since $D_q(a_1, y) = n - 1$ and $D_q(b_2, y) = n - 2$ it follows by induction that

$$N(x, y) = |A| + |B| \leq N(a_1, y) + N(b_2, y) \leq F_{n-1} + F_{n-2} = F_n.$$

This completes the proof of the first assertion of the theorem.

To prove the second assertion of the theorem, that the bound F_n can be attained when q is even, we illustrate in Figure 2.14 q -chains (with $q = 2r$ for some integer r) that attain the bound, and highlight the $q = 4$ case. The labels in Figure 2.14 give the number of edges between pairs of vertices. That these examples do indeed attain the bound can be proved using a simple induction argument similar to that above; the details are omitted here. \square

The proof of the first assertion of Theorem 2.17 holds for vertices x and y of \mathcal{F}_q with q odd; these bounds are not, however, the best possible in this case. The following theorem gives the best possible bounds on N when x and y are vertices of \mathcal{F}_q with q odd.

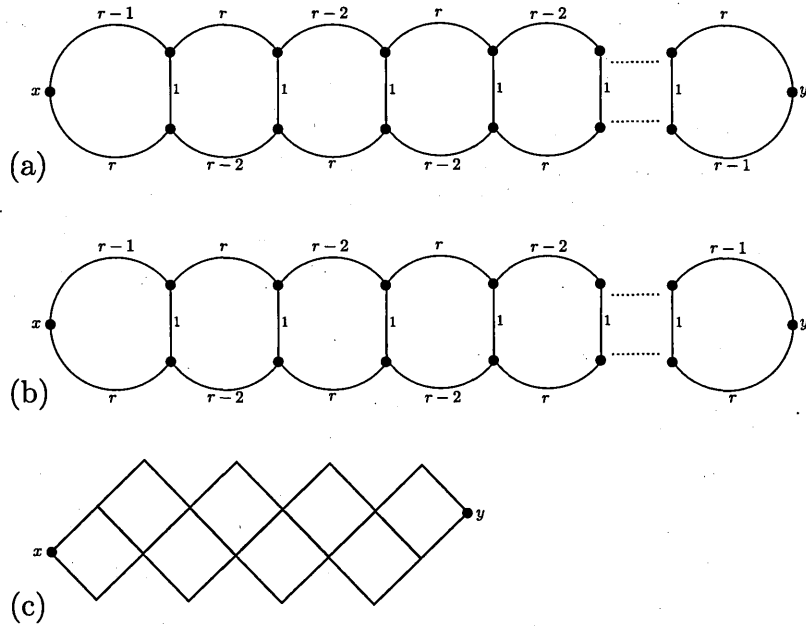


Figure 2.14: (a) A $2r$ -chain for which $N(x, y) = F_n$ (n even) (b) A $2r$ -chain for which $N(x, y) = F_n$ (n odd) (c) A 4-chain for which $N(x, y) = F_n$

Theorem 2.18. Suppose that x and y are vertices of \mathcal{F}_q , for $q > 3$ odd, with $D_q(x, y) = n$. Then

$$N(x, y) \leq \begin{cases} F_{n/2}, & n \text{ even}, \\ 2F_{(n-3)/2}, & n \text{ odd}. \end{cases}$$

Furthermore, there are vertices x and y with $D_q(x, y) = n$ for which these bounds can be attained.

Proof. We prove the inequalities by using induction on n . When $n = 0$ or $n = 1$ it is immediate that $N(x, y) = 1$. When n is 2 or 3, simple case analysis shows that there are at most two geodesic paths between x and y ; the two possibilities when n is 2 and $q = 2r + 1$ for some integer r are illustrated in Figure 2.15.

Suppose now that $n > 3$, and assume that the inequalities are true for all pairs of vertices u and v with $D_q(u, v)$ less than n . Choose two vertices x and y with $D_q(x, y) = n$, and let P_1, \dots, P_n be the q -chain

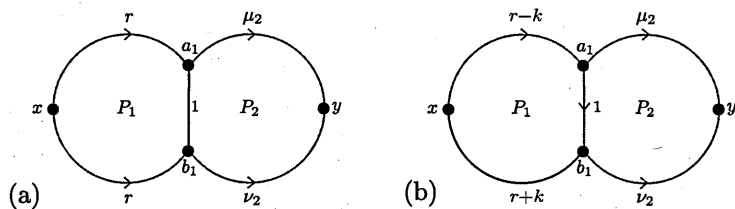


Figure 2.15: (a) A $(2r+1)$ -chain in which $|\mu_1| = |\nu_1| = r$; there are at most two geodesic paths between x and y (b) A $(2r+1)$ -chain in which $|\mu_1| < |\nu_1|$; again, there are at most two geodesic paths between x and y

from x to y , illustrated in Figure 2.13. Without loss of generality, we may assume that $|\mu_1| \leq |\nu_1|$.

Suppose first that n is even. If $|\mu_1| < |\nu_1|$, then $|\mu_1| \leq \lfloor n/2 \rfloor - 2$ and every geodesic path must travel along the path μ_1 . Since $D_q(a_1, y) = n - 1$ we see by induction that

$$N(x, y) = N(a_1, y) \leq 2F_{(n-4)/2} < F_{n/2}.$$

The remaining possibility is that $|\mu_1| = |\nu_1|$. Then no geodesic path from x to y may pass through both a_1 and b_1 , and so we can partition the set of geodesic paths from x to y into the set A of those geodesic paths that travel along the path μ_1 followed by the path μ_2 and the set B of those geodesic paths that travel along the path ν_1 followed by the path ν_2 . Since $|\mu_1| = |\nu_1|$, we can assume without loss of generality that $|\mu_2| \leq |\nu_2|$. We now have two possibilities.

Firstly, suppose that $|\mu_2| < |\nu_2| - 1$. Then every geodesic path from x to y must pass along the path μ_1 followed by the path μ_2 . Then $B = \emptyset$ and since $D_q(a_2, y) = n - 2$ we see that

$$N(x, y) = N(a_2, y) \leq F_{(n-2)/2} < F_{n/2}.$$

Secondly, suppose that $|\mu_2| = |\nu_2| - 1$. Now no path in B may pass through a_3 , and so every path in B must pass along the consecutive paths ν_1 , ν_2 and ν_3 . We now consider the possibilities for $|\mu_3|$ and

$|v_3|$. If $|\mu_3| < |v_3|$ then every geodesic path from x to y must pass along the consecutive paths μ_1, μ_2 and μ_3 . Then $B = \emptyset$ and since $D_q(a_3, y) = n - 3$ we see that

$$N(x, y) = N(a_3, y) \leq 2F_{(n-6)/2} < F_{n/2}.$$

If $|\mu_3| > |v_3| + 1$ then all paths in set A will pass through both a_2 and b_2 from where they must pass along the path v_3 to b_3 . Since $D_q(b_3, y) = n - 3$ we see that

$$N(x, y) = |A| + |B| = 2N(b_3, y) \leq 4F_{(n-6)/2} < F_{n/2}.$$

Finally, suppose that $|\mu_3| = |v_3| + 1$. No geodesic path from x to y may pass through both a_3 and b_3 . It follows that every path in B must pass along the consecutive paths v_1, v_2, v_3 and v_4 . Paths in A , however, may pass along either consecutive paths μ_1, μ_2, μ_3 and μ_4 or the consecutive paths $\mu_1, \mu_2, \langle a_2, b_2 \rangle, v_3$ and v_4 . If $n = 4$, then simple case analysis shows that there are at most $3 = F_{n/2}$ geodesic paths between x and y . Otherwise, we consider the possibilities for $|\mu_4|$ and $|v_4|$.

If $|\mu_4| < |v_4| - 1$ then every geodesic path from x to y must pass along the consecutive paths μ_1, μ_2, μ_3 and μ_4 , and since $D_q(a_4, y) = n - 4$,

$$N(x, y) = N(a_4, y) \leq F_{(n-4)/2} < F_{n/2}.$$

If $|\mu_4| > |v_4| + 1$ then every path in A must pass along the consecutive paths $\mu_1, \mu_2, \langle a_2, b_2 \rangle, v_3$ and v_4 , and since $D_q(b_4, y) = n - 4$ we see that

$$N(x, y) = |A| + |B| = 2N(b_4, y) \leq 2F_{(n-4)/2} < F_{n/2}.$$

If $|\mu_4| = |\nu_4| - 1$ then no geodesic path between x and y may pass from b_4 to a_4 . This means that every path in B must pass along the consecutive paths $\nu_1, \nu_2, \nu_3, \nu_4$ and ν_5 . We partition paths in A into the set A_1 of those paths that travel along the consecutive paths μ_1, μ_2, μ_3 , and μ_4 and the set A_2 of those paths that travel along the consecutive paths $\mu_1, \mu_2, \langle a_2, b_2 \rangle, \nu_3, \nu_4$ and ν_5 . Since $D_q(a_4, y) = n - 4$ and $D_q(b_5, y) = n - 5$, we see that

$$\begin{aligned} N(x, y) &= |A_1| + |A_2| + |B| \\ &= N(a_4, y) + 2N(b_5, y) \\ &\leq F_{(n-4)/2} + 4F_{(n-8)/2} \\ &< F_{n/2}. \end{aligned}$$

Finally, we consider the case in which $|\mu_4| = |\nu_4| + 1$. Again, we partition A into the sets A_1 and A_2 described above. Now no geodesic path between x and y may travel through a_4 and then b_4 . We consider the possibilities for $|\mu_5|$ and $|\nu_5|$. If $|\mu_5| > |\nu_5|$ then $A_1 = \emptyset$ and all paths in A_2 and B must travel from b_4 to b_5 . Since $D_q(b_5, y) = n - 5$, it follows that

$$N(x, y) = |A_2| + |B| = 2N(b_5, y) = 2F_{(n-8)/2} < F_{n/2}.$$

If $|\mu_5| < |\nu_5| - 1$ then both A_2 and B are empty and so

$$N(x, y) = |A_1| = N(a_4, y) = F_{(n-4)/2} < F_{n/2}.$$

If $|\mu_5| = |\nu_5| - 1$ then no geodesic path between x and y may pass through both a_5 and b_4 , and so every path in A_1 must travel along the

consecutive paths $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and ν_6 . Since $D_q(a_6, y) = n - 6$, we see that

$$\begin{aligned} N(x, y) &= |A_1| + |A_2| + |B| \\ &= N(a_6, y) + 2N(b_4, y) \\ &\leq F_{(n-6)/2} + 2F_{(n-4)/2} \\ &= F_{n/2}. \end{aligned}$$

This proves the first assertion of the theorem when n is even. The proof that $N(x, y) \leq 2F_{(n-3)/2}$ when n is odd follows in exactly the same fashion; the details are omitted here.

To prove the second assertion of the theorem, that the bounds stated can be attained, we illustrate in Figure 2.16 q -chains (with $q = 2r + 1$ for some integer r) that attain the bound. The labels in Figure 2.16 give the number of edges between pairs of vertices. That these examples do indeed attain the bounds can be proved using a simple induction argument; the details are omitted here. \square

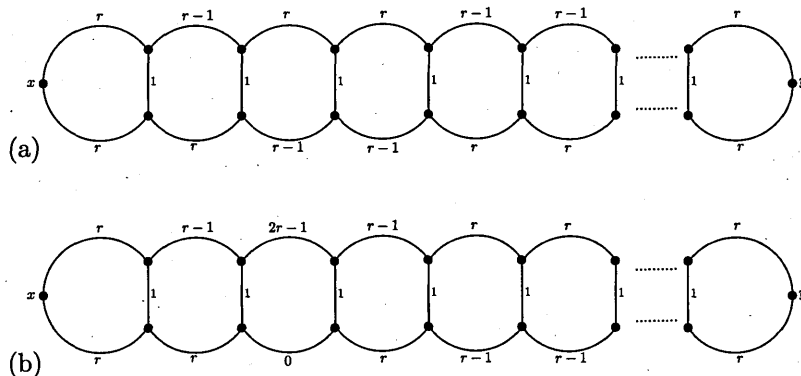


Figure 2.16: (a) A $(2r + 1)$ -chain for which $N(x, y) = F_{n/2}$ (n even) (b) A $(2r + 1)$ -chain for which $N(x, y) = F_{(n-3)/2}$ (n odd)

When $q = 3$ the bounds given in Theorem 2.18 hold; when n is odd, however, there is a better bound $N(x, y) \leq F_{(n-3)/2}$.

Theorem 2.19. Suppose that x and y are vertices of \mathcal{F}_3 , with $D_3(x, y) = n$.

Then

$$N(x, y) \leq \begin{cases} F_{n/2}, & n \text{ even}, \\ F_{(n-3)/2}, & n \text{ odd}. \end{cases}$$

Furthermore, there are vertices x and y with $D_3(x, y) = n$ for which these bounds can be attained.

Proof. We prove the inequalities by using induction on n . They are immediate if $n = 0$ or $n = 1$. Suppose now that $n > 1$, and assume that the inequalities are true for all pairs of vertices u and v with $D_q(u, v) < n$. Choose two vertices x and y with $D_q(x, y) = n$, and let P_1, \dots, P_n be the q -chain from x to y , illustrated in Figure 2.13. Note that for each i , the lengths $|\mu_i|$ and $|v_i|$ are either 0 or 1, and we must have $|\mu_1| = |v_1| = 1$ and $|\mu_n| = |v_n| = 1$. If $n = 2$, it is therefore clear that $N(x, y) = 2 = F_{n/2}$. Else we can assume without loss of generality that $|\mu_2| \leq |v_2|$, and so $|\mu_2| = 0$ and $|v_2| = 1$.

If $|\mu_3| = 0$ and $|v_3| = 1$, or if $n = 3$ and $|\mu_3| = |v_3| = 1$, then every geodesic path from x to y must pass along the path μ_1 . Since $a_1 = a_2$ and $D_q(a_2, y) = n - 2$ we see by induction that if n is even

$$N(x, y) = N(a_2, y) \leq F_{(n-2)/2} < F_{n/2},$$

and if n is odd

$$N(x, y) = N(a_2, y) \leq F_{(n-5)/2} < F_{(n-3)/2}.$$

Else $|\mu_3| = 1$ and $|v_3| = 0$ and our 3-chain is as shown in Figure 2.17. We partition the set of geodesic paths from x to y into the set A of those geodesic paths that travel along the path μ_1 and the set B of those geodesic paths that travel along the path v_1 . A path in A cannot

pass through b_1 as well as $a_1 = a_2$ because it is a geodesic path and $|v_1| < |\mu_1| + 1$. Similarly, a path in B cannot pass through $a_1 = a_2$ as well as b_1 ; instead it must pass through $b_2 = b_3$. Furthermore, a path in B may not travel through a_3 as it is a geodesic path and $|\mu_1| + |\mu_2| + |\mu_3| < |v_1| + |v_2| + 1$. Instead it must pass through b_4 . Since $D_q(a_2, y) = n - 2$ and $D_q(b_4, y) = n - 4$ it follows by induction that if n is even,

$$\begin{aligned}
 N(x, y) &= |A| + |B| \\
 &\leq N(a_2, y) + N(b_4, y) \\
 &\leq F_{(n-2)/2} + F_{(n-4)/2} \\
 &= F_{n/2}.
 \end{aligned}$$

The case in which n is odd follows in exactly the same fashion; we omit the details here. This completes the proof of the first assertion of the theorem.

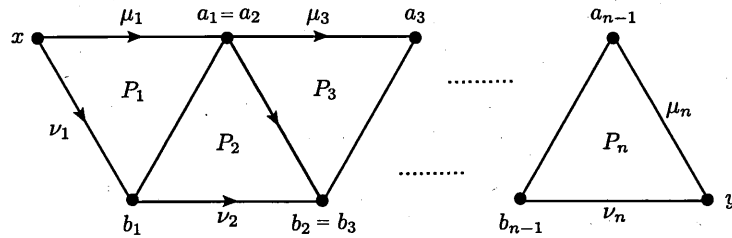


Figure 2.17: A 3-chain in which $|\mu_1| = |v_1| = 1$, $|\mu_2| = 0$, $|v_2| = 1$, $|\mu_3| = 1$ and $|v_3| = 0$.

To prove the second assertion of the theorem, we illustrate in Figure 2.18 3-chains that attain the bounds $N(x, y) \leq F_{n/2}$ for even n and $N(x, y) \leq 2F_{(n-3)/2}$ for odd n . That these examples do indeed attain the bound can be proved using a simple induction argument similar to that above; the details are omitted here. \square

Theorem 2.17 and Theorem 2.18 give upper bounds for the function N ; the lower bound for N , no matter the value of $D_q(x, y)$, is 1. For

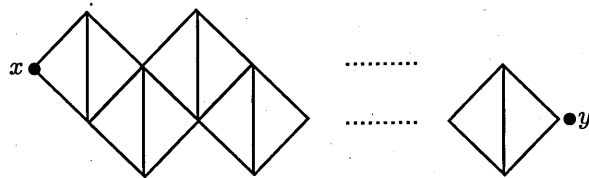


Figure 2.18: A 3-chain for which $N(x, y) = F_{n/2}$ for even n . The case when n is odd is similar.

example, let $x = \infty$ and $y = [0, n]_q$. Then there is a unique geodesic path between x and y , namely $\langle x, 0, y \rangle$, with $D_q(x, y) = n$.

2.3.3 NUMBER OF GEODESIC ROSEN CONTINUED FRACTIONS IN TERMS OF $d_q(\infty, x)$

Recall that given two vertices x and y of \mathcal{F}_q , $d_q(x, y)$ is the length of a geodesic path between x and y ; $d_q(\infty, y)$ is therefore the length of a geodesic Rosen continued fraction expansion of y . In this section we give upper bounds on the number of geodesic Rosen continued fraction expansions of a vertex y of \mathcal{F}_q in terms of $n = d_q(\infty, y)$. We do so by giving an upper bound on the number of geodesic paths in \mathcal{F}_q between two vertices x and y with $d_q(x, y) = n$. Although these bounds are more complicated than those given in Theorem 2.17 and Theorem 2.18, they give greater insight into the number of geodesic Rosen continued fraction expansions a vertex may have; we discuss this towards the end of the section.

Recall that we denote by F_i the i th term of the Fibonacci sequence given by $F_{-1} = 1$, $F_0 = 1$, $F_1 = 2$, $F_2 = 3$, $F_3 = 5$, and so forth, and that given vertices x and y of \mathcal{F}_q , $N(x, y)$ denotes the number of geodesic paths in \mathcal{F}_q between x and y . When $q = 3$ it is shown in [3] that for a vertex y of \mathcal{F}_3 with $d_q(\infty, y) = n > 0$, there are at most F_{n-1} geodesic Rosen continued fraction expansions of y , and this bound is attained. The case $q = 3$ is exceptional in that the bound on

N when $q = 3$ cannot be deduced from the bound on N for general q . The same is true when $q = 4$ or when $q = 6$; we begin by discussing these two cases.

Theorem 2.20. *Suppose that x and y are vertices of \mathcal{F}_4 with $d_q(x, y) = n$. Then $N(x, y) \leq F_{n-1}$. Furthermore, there are vertices x and y with $d_q(x, y) = n$ for which these bounds can be attained.*

Proof. We prove the inequality $N(x, y) \leq F_{n-1}$ by using induction on n . It is immediate if n is 0 or 1. Suppose now that $n > 1$, and assume that the inequality is true for all pairs of vertices u and v with $d_q(u, v) < n$. Choose two vertices x and y with $d_q(x, y) = n$, and let P_1, \dots, P_m be the q -chain from x to y , illustrated in Figure 2.13. Without loss of generality, we may assume that $|\mu_1| \leq |v_1|$, so that $|\mu_1| = 1$ and $|v_1| = 2$. We partition the set of geodesic paths from x to y into the set A of those geodesic paths that travel along the path μ_1 and the set B of those geodesic paths that travel along the path v_1 . A path in B cannot pass through a_1 as well as b_1 because it is a geodesic path and $|\mu_1| < |v_1| + 1$. Instead it must pass through b_2 . Since $D_q(a_1, y) = n - 1$ and $D_q(b_2, y) = n - 2$ it follows by induction that

$$\begin{aligned} N(x, y) &= |A| + |B| \\ &\leq N(a_1, y) + N(b_2, y) \\ &\leq F_{n-2} + F_{n-3} \\ &= F_{n-1}. \end{aligned}$$

This completes the proof of the first assertion of the theorem.

To prove the second assertion of the theorem, we illustrate in Figure 2.19 a 4-chain admitting a geodesic path that attains the bound $N(x, y) \leq F_{n-1}$ for all n . That this path does indeed attain the bound

can be proved using a simple induction; the details are omitted here.

□

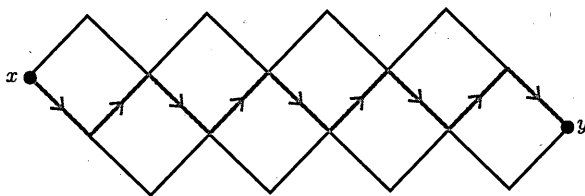


Figure 2.19: A 4-chain containing a geodesic path for which $N(x, y) = F_{n-1}$ for all n

Theorem 2.21. Suppose that x and y are vertices of \mathcal{F}_6 with $d_q(x, y) = n$.

Then

$$N(x, y) \leq \begin{cases} 1, & n \leq 2, \\ 2F_{(n-4)/2}, & n \text{ even}, n > 2, \\ F_{(n-1)/2}, & n \text{ odd}, n > 2. \end{cases}$$

Furthermore, there are vertices x and y with $d_q(x, y) = n$ for which these bounds can be attained.

Proof. The proof of the first assertion of the theorem is similar to the proofs of Theorem 2.18 and Theorem 2.20. To prove the second assertion of the theorem, we illustrate in Figure 2.20 6-chains admitting geodesic paths that attain the bounds $N(x, y) \leq 2F_{(n-4)/2}$ for even n and $N(x, y) \leq F_{(n-1)/2}$ for odd n when $n > 2$. That this path does indeed attain the bound can be proved using a simple induction; the details are omitted here. □

We now consider the general case in which q is even and greater than 6. Write $q = 2r$ and let $[x]$ denote the integer part of x . We define a function H on \mathbb{N} as follows.

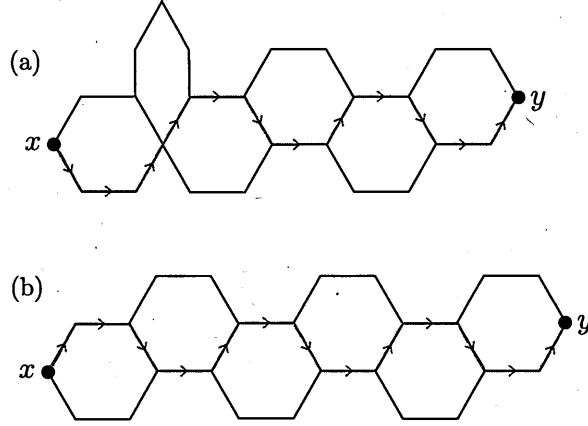


Figure 2.20: (a) A 6-chain containing a geodesic path for which $N(x,y) = 2F_{(n-4)/2}$ for all even n (b) A 6-chain containing a path for which $N(x,y) = F_{(n-1)/2}$ for all odd n

$$H(n) = \begin{cases} F_{[(n-1)/(r-1)]}, & 0 \leq n \leq q-1, \\ 2^{\lfloor n/r \rfloor}, & n \geq q \text{ and } n \equiv 0, \dots, r-3 \pmod{r}, \\ 5 \times 2^{\lfloor n/r \rfloor - 2}, & n \geq q \text{ and } n \equiv r-2 \pmod{r}, \\ 3 \times 2^{\lfloor n/r \rfloor - 1}, & n \geq q \text{ and } n \equiv r-1 \pmod{r}. \end{cases}$$

Notice that H is an increasing function.

Theorem 2.22. Suppose x and y are vertices of \mathcal{F}_q , where $q > 6$ is even, and $d_q(x,y) = n$. Then $N(x,y) \leq H(n)$. Furthermore, there are vertices x and y with $d_q(x,y) = n$ for which these bounds can be attained.

Proof. If n is 0 or 1 then the result is clear. If $1 < n < q$, then the result follows from simple case analysis. For $n > q-1$ we use induction on n . If $q \leq n < 3r$ then the result follows from simple case analysis. Suppose now that $n \geq 3r$, and assume that $N(x,y) \leq H(n)$ for all pairs of vertices u and v with $d_q(x,y) < n$. Choose two vertices x and y with $d_q(x,y) = n$, and let P_1, \dots, P_m be the q -chain from x to y , illustrated in Figure 2.13. Without loss of generality, we may assume that $|\mu_1| \leq |\nu_1|$. If $|\mu_1| \leq |\nu_1| - 1$, then every geodesic path from x to

y must pass along the path μ_1 . Since $|\mu_1| \geq 1$, $d_q(a_1, y) \leq n-1$, and we see by induction that

$$N(x, y) = N(a_1, y) \leq H(n-1) < H(n),$$

since the function H is strictly increasing for $n \geq q$. Otherwise, $|\mu_1| = |v_1| - 1$, so that $|\mu_1| = r-1$ and $|v_1| = r$.

If $|\mu_2| < |v_2|$ then every geodesic path from x to y must pass along the path μ_1 directly followed by the path μ_2 . Since $d_q(a_2, y) \leq n-r+1$, we see by induction that

$$N(x, y) = N(a_2, y) \leq H(n-r+1) < H(n).$$

For all remaining cases, the set of geodesic paths from x to y can be partitioned into the set A of those geodesic paths that travel along μ_1 and the set B of those geodesic paths that travel along the path v_1 . A path in B cannot pass through a_1 as well as b_1 because it is a geodesic path and $|\mu_1| < |v_1| + 1$. Instead it must pass through b_2 .

If $|\mu_2| > |v_2| + 2$ then every path in A must pass through b_1 and travel along the path v_2 . Since $d_q(b_2, y) \leq n-r$, we see by induction that

$$N(x, y) = |A| + |B| \leq 2H(n-r) < H(n),$$

where this latter inequality follows immediately because $n-r \equiv n \pmod{r}$.

The remaining possibilities are that $|\mu_2| = |v_2|$ or $|\mu_2| = |v_2| + 2$. In either case, $|v_2| \geq r-2$, and so $d_q(b_2, y) \leq n-q+2$. Since $d_q(a_1, y) = n-r+1$, we see by induction that

$$N(x, y) = |A| + |B| \leq H(n-r+1) + H(n-q+2).$$

$n \pmod r$	$H(n-r+1)$	$H(n-q+2)$	$H(n)$
$0, \dots, r-5$	$2^{\lfloor n/r \rfloor - 1}$	$2^{\lfloor n/r \rfloor - 2}$	$2^{\lfloor n/r \rfloor}$
$r-4$	$2^{\lfloor n/r \rfloor - 1}$	$5 \times 2^{\lfloor n/r \rfloor - 4}$	$2^{\lfloor n/r \rfloor}$
$r-3$	$5 \times 2^{\lfloor n/r \rfloor - 3}$	$3 \times 2^{\lfloor n/r \rfloor - 3}$	$2^{\lfloor n/r \rfloor}$
$r-2$	$3 \times 2^{\lfloor n/r \rfloor - 2}$	$2^{\lfloor n/r \rfloor - 1}$	$5 \times 2^{\lfloor n/r \rfloor - 2}$
$r-1$	$2^{\lfloor n/r \rfloor}$	$2^{\lfloor n/r \rfloor - 1}$	$3 \times 2^{\lfloor n/r \rfloor - 1}$

Table 1: Values of $H(n-r+1)$, $H(n-q+2)$ and $H(n)$ for different values of $n \pmod r$.

The value of $H(n-r+1) + H(n-q+2)$ is dependent on the value of $n \pmod r$. We summarise the possibilities in Table 1. In all cases, we see that $H(n-r+1) + H(n-q+2) \leq H(n)$. This completes the proof of the first assertion of the theorem.

To prove the second assertion of the theorem, we illustrate in Figure 2.21 q -chains for $q = 2r$ admitting geodesic paths that attain the bound $N(x, y) \leq H(n)$. The labels in Figure 2.21 give the number of edges between pairs of vertices. That these paths do indeed attain the bound can be proved using a simple induction; the details are omitted here. \square

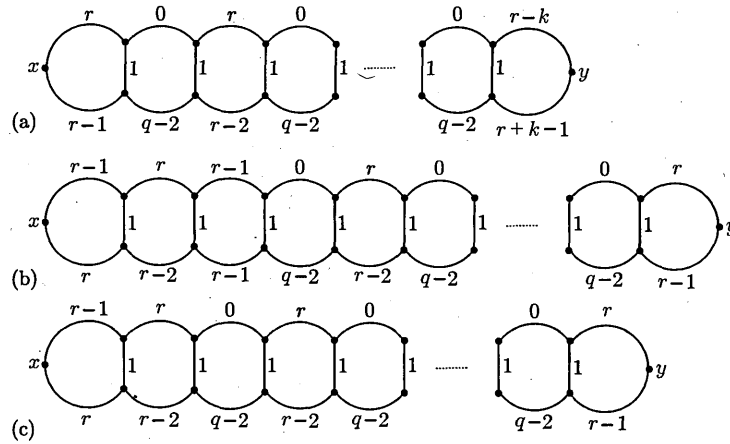


Figure 2.21: (a) A $2r$ -chain containing a geodesic path for which $N(x, y) = 2^{\lfloor n/r \rfloor}$, where k is such that $n \equiv 0, \dots, r-3 \pmod r$ (b) A $2r$ -chain containing a geodesic path for which $N(x, y) = 5 \times 2^{\lfloor n/r \rfloor - 2}$ for all $n \equiv r-2 \pmod r$ (c) A $2r$ -chain containing a geodesic path for which $N(x, y) = 3 \times 2^{\lfloor n/r \rfloor - 1}$ for all $n \equiv r-1 \pmod r$

We finally consider the case in which q is odd and greater than 3.

Theorem 2.23. *Suppose x and y are vertices of \mathcal{F}_q , where $q > 3$ is odd, and $d_q(x, y) = n$. Then*

$$N(x, y) \leq \begin{cases} F_{[(n-1)/(q-2)]}, & 1 \leq n \leq 2q-3, \\ 2^{\lfloor n/(q-1) \rfloor}, & n \geq q \text{ and } n \equiv 0, \dots, q-4 \pmod{q-1}, \\ 5 \times 2^{\lfloor n/(q-1) \rfloor - 2}, & n \geq q \text{ and } n \equiv q-3 \pmod{q-1}, \\ 3 \times 2^{\lfloor n/(q-1) \rfloor - 1}, & n \geq q \text{ and } n \equiv q-2 \pmod{q-1}. \end{cases}$$

Furthermore, there are vertices x and y with $d_q(x, y) = n$ for which these bounds can be attained.

Proof. We omit the proof of the first assertion of the theorem, as it is similar to the proofs of Theorem 2.18 and Theorem 2.22. To prove the second assertion of the theorem, we illustrate in Figure 2.21 q -chains for $q = 2r + 1$ admitting geodesic paths that attain the bound $N(x, y) \leq H(n)$. The labels in Figure 2.21 give the number of edges between pairs of vertices. That these paths do indeed attain the bound can be proved using a simple induction; the details are omitted here. \square

Together, Theorems 2.18, 2.20, 2.21 and 2.22 give upper bounds for the function N . We saw in Section 2.3.2 that for any n , there exist vertices x and y of \mathcal{F}_q such that $D_q(x, y) = n$ but $N(x, y) = 1$. It is also the case that for any n there are x and y such that $d_q(x, y) = n$ but $N(x, y) = 1$. For example, let $x = \infty$ and $y = [0, 2, 2, 2, \dots, 2]_q$ where the number of 2s is $n - 1$; $d_q(x, y) = n$. The q -chain from x to y is illustrated in Figure 2.23, with the the labels giving the number of edges between pairs of vertices. It is clear that there is a unique geodesic path between x and y , and so $N(x, y) = 1$.

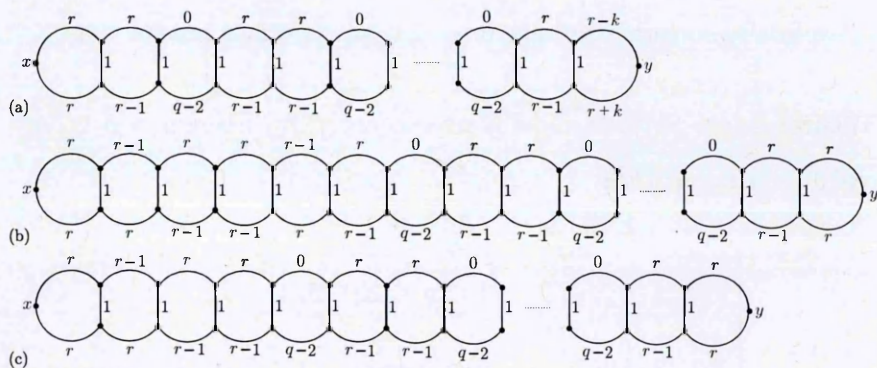


Figure 2.22: (a) A $(2r+1)$ -chain containing a geodesic path for which $N(x, y) = 2^{\lfloor n/(q-1) \rfloor}$ where k is such that $n \equiv 0, 1, \dots, q-4 \pmod{q-1}$ (b) A $(2r+1)$ -chain containing a geodesic path for which $N(x, y) = 5 \times 2^{\lfloor n/(q-1) \rfloor - 2}$ for all $n \equiv q-3 \pmod{q-1}$ (c) A $(2r+1)$ -chain containing a geodesic path for which $N(x, y) = 3 \times 2^{\lfloor n/(q-1) \rfloor - 1}$ for all $n \equiv q-2 \pmod{q-1}$

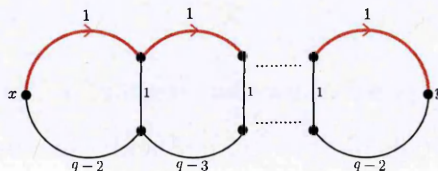


Figure 2.23: A path between two vertices x and y of \mathcal{F}_q with $N(x, y) = 1$

We conclude this section with a brief comparison of the bounds on the number of geodesic Rosen continued fraction expansions of a vertex of \mathcal{F}_q that we have obtained in this section and in [Section 2.3.2](#). Consider a vertex y of \mathcal{F}_q . The theorems giving the upper bound on $N(\infty, y)$ in terms of $D_q(\infty, y)$ are relatively simple when compared to the theorems giving the upper bound on $N(\infty, y)$ in terms of $d_q(\infty, y)$. The quantity $D_q(\infty, y)$, however, is not as straightforward to compute as $d_q(\infty, y)$, which is simply the length of a geodesic Rosen continued fraction expansion of y (take, for example, the Rosen continued fraction expansion of y given by the nearest-integer algorithm). Furthermore, the bound given in terms of $D_q(\infty, y)$ is generally larger than the bound given in terms of $d_q(\infty, y)$. We illustrate this claim in the case $q = 5$.

Given a vertex y of \mathcal{F}_5 , let $m = D_5(\infty, y)$ and let $n = d_5(\infty, y)$.

Notice that we must have

$$n \leq 4 + \left\lceil \frac{3(m-2)}{2} \right\rceil,$$

since both ∞ and y lie on the perimeter of the q -chain between ∞ and y , and the path traversing this perimeter has length $8 + 3(m-2)$. Let N_D denote the maximum number of geodesic Rosen continued fraction expansions of y in terms of $D_5(\infty, y)$, as given by Theorem 2.18, and let N_d denote the maximum number of geodesic Rosen continued fraction expansions of y in terms of $d_5(\infty, y)$, as given by Theorem 2.23. The values $N_D - N_d$ for a range of values of $D_5(\infty, y)$ and $d_5(\infty, y)$ are shown in Table 2; notice that in almost all cases, $N_D > N_d$, and hence the bound in terms of $d_q(\infty, y)$ is in general a better bound than that given in terms of $D_q(\infty, y)$.

Finally, we note that the bound on the function N in terms of $D_q(x, y)$ is independent of q ; this is not the case for the bound on N in terms of $d_q(x, y)$. By considering the bound on N in terms of $d_q(x, y)$, we see that for any fixed n there is sufficiently large q so that for any y with $d_q(\infty, y) = n$, $N(\infty, y) = 1$. As q tends towards ∞ , λ_q tends towards $\lambda_\infty = 2$. We might predict, therefore, that if a continued fraction of the form

$$b_1\lambda_\infty - \frac{1}{b_2\lambda_\infty - \cdots - \frac{1}{b_n\lambda_\infty}},$$

is a geodesic continued fraction expansion of a real number y , then it is the only geodesic continued fraction expansion of y of this form. We see in Chapter 3 that this is indeed the case; in fact, $[b_1, \dots, b_n]_\infty$ is the only continued fraction expansion of y of this form.

$d_5(\infty, y)$	$D_5(\infty, y)$										
	5	6	7	8	9	10	11	12	13	14	15
5	2	3	4	6	8	11	14	19	24	32	40
6	2	3	4	6	8	11	14	19	24	32	40
7	1	2	3	5	7	10	13	18	23	31	39
8	0	1	2	4	6	9	12	17	22	30	38
9	-	1	2	4	6	9	12	17	22	30	38
10	-	0	1	3	5	8	11	16	21	29	37
11	-	-	0	2	4	7	10	15	20	28	36
12	-	-	-	0	2	5	8	13	18	26	34
13	-	-	-	0	2	5	8	13	18	26	34
14	-	-	-	-	0	3	6	11	16	24	32
15	-	-	-	-	-	1	4	9	14	22	30
16	-	-	-	-	-	-3	0	5	10	18	26
17	-	-	-	-	-	-	0	5	10	18	26
18	-	-	-	-	-	-	-	1	6	14	22
19	-	-	-	-	-	-	-	-3	2	10	18
20	-	-	-	-	-	-	-	-	-6	2	10
21	-	-	-	-	-	-	-	-	-	2	10
22	-	-	-	-	-	-	-	-	-	-6	2
23	-	-	-	-	-	-	-	-	-	-	-6

Table 2: The difference $N_D - N_d$ for different values of $D_5(\infty, y)$ and $d_5(\infty, y)$

2.4 CHARACTERISING GEODESIC ROSEN CONTINUED FRACTIONS

Our next pair of theorems give necessary and sufficient conditions for $[b_1, \dots, b_n]_q$ to be a geodesic Rosen continued fraction. We give separate results for when q is odd and when q is even. To formulate our result concisely, we use the notation $1^{[d]}$ to mean the sequence consisting of d consecutive 1s. Also, given a sequence x_1, \dots, x_n , we write $\pm(x_1, \dots, x_n)$ to mean one of the two sequences x_1, \dots, x_n or $-x_1, \dots, -x_n$.

Theorem 2.24. *Suppose that $q = 2r$, where $r \geq 2$. The continued fraction $[b_1, \dots, b_n]_q$ is a geodesic Rosen continued fraction if and only if the se-*

quence b_2, \dots, b_n has no terms equal to 0 and contains no subsequence of consecutive terms either of the form $\pm 1^{[r]}$ or of the form

$$\pm(1^{[r-1]}, 2, 1^{[r-2]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-1]}).$$

Theorem 2.25. Suppose that $q = 2r + 1$, where $r \geq 2$. The continued fraction $[b_1, \dots, b_n]_q$ is a geodesic Rosen continued fraction if and only if the sequence b_2, \dots, b_n has no terms equal to 0 and contains no subsequence of consecutive terms either of the form $\pm 1^{[r]}$ or of the form

$$\pm(1^{[d_1]}, 2, 1^{[d_2]}, 2, \dots, 1^{[d_k]}),$$

where k is odd, at least 3, and d_1, \dots, d_k is a sequence of the form

$$r-1, \underbrace{r-1, r-2, r-1, r-2, \dots, r-1}_{\text{alternating } r-1 \text{ and } r-2}, r-1.$$

The number of 2s in the sequences above may be any positive integer.

The version of Theorem 2.25 is a little different when $q = 3$, and has already been established in [3, Theorem 1.2], so we omit it here.

In geometric terms, Theorems 2.24 and 2.25 say that the path corresponding to a Rosen continued fraction is a geodesic path unless it either doubles back on itself or takes “the long way round” the outside of a chain of q -gon. These possibilities are illustrated when $q = 4$ in Figure 2.24 and when $q = 5$ in Figure 2.25.

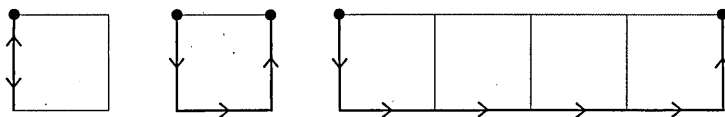


Figure 2.24: A path in \mathcal{F}_4 that is not a geodesic path must contain a subpath of type similar to one of these.

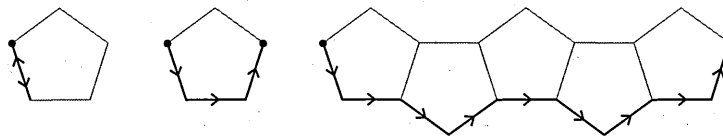


Figure 2.25: A path in \mathcal{F}_5 that is not a geodesic path must contain a subpath of type similar to one of these.

2.4.1 PROOF OF THEOREM 2.24 AND THEOREM 2.25: PART I

In this section we prove one of the implications from each of Theorems 2.24 and 2.25 (the implications that say that if b_2, \dots, b_n contains a subsequence of a certain type, then $[b_1, \dots, b_n]_q$ is not a geodesic Rosen continued fraction), leaving the other implications for Section 2.4.2.

This part of the two proofs relies on particular relations from the Hecke group Γ_q . Recall that $\sigma(z) = -1/z$, $\tau(z) = z + \lambda_q$, and $\kappa(z) = -\bar{z}$. For each integer b , let

$$T_b(z) = b\lambda_q - \frac{1}{z}.$$

That is, $T_b = \tau^b \sigma$. Observe that $\kappa T_b \kappa = T_{-b}$. This has the following useful consequence.

Lemma 2.26. *The continued fraction $[b_1, \dots, b_n]_q$ is a geodesic Rosen continued fraction if and only if $[-b_1, \dots, -b_n]_q$ is a geodesic Rosen continued fraction.*

Proof. Let $\gamma = \langle \infty, v_1, \dots, v_n \rangle$ be the path of convergents of the Rosen continued fraction $[b_1, \dots, b_n]_q$ and let $\delta = \langle \infty, w_1, \dots, w_n \rangle$ be the

path of convergents of the Rosen continued fraction $[-b_1, \dots, -b_n]_q$.

Then

$$\begin{aligned} w_i &= T_{-b_1} \cdots T_{-b_i}(\infty) \\ &= (\kappa T_{b_1} \kappa) \cdots (\kappa T_{b_n} \kappa)(\infty) \\ &= \kappa T_{b_1} \cdots T_{b_n}(\infty) \\ &= \kappa(v_i). \end{aligned}$$

Therefore δ is the image of γ under the automorphism κ of \mathcal{F}_q . It follows that γ is a geodesic path if and only if δ is a geodesic path. \square

The next two lemmas contain useful identities. Recall that I is the identity element of Γ_q .

Lemma 2.27. *Let $q = 2r$. Then $T_1^r = \sigma\tau^{-1}\sigma T_{-1}^{r-2}\tau^{-1}$.*

Proof. Since $\sigma^2 = I$, it is straightforward to check that the identity is equivalent to the relation $(\tau\sigma)^{2r} = I$. \square

Lemma 2.28. *Let $q = 2r$. Then for each integer $k = 0, 1, 2, \dots$,*

$$T_1^{r-1}(T_2 T_1^{r-2})^k T_2 T_1^{r-1} = \sigma\tau^{-1}\sigma(T_{-1}^{r-2} T_{-2})^{k+1} T_{-1}^{r-2}\tau^{-1}.$$

Proof. Using the relation $(\tau\sigma)^{2r} = I$, we can check that

$$T_2 T_1^{r-2} = (\tau\sigma\tau^{-1}\sigma)(T_{-1}^{r-2} T_{-2})(\tau\sigma\tau^{-1}\sigma)^{-1}.$$

Therefore

$$\begin{aligned}
 & T_1^{r-1} (T_2 T_1^{r-2})^k T_2 T_1^{r-1} \\
 &= T_1 T_2^{-1} (T_2 T_1^{r-2})^{k+2} T_1 \\
 &= \tau^{-1} (T_2 T_1^{r-2})^{k+2} \tau \sigma \\
 &= \tau^{-1} (\tau \sigma \tau^{-1} \sigma) (T_{-1}^{r-2} T_{-2})^{k+2} (\tau \sigma \tau^{-1} \sigma)^{-1} \tau \sigma \\
 &= (\sigma \tau^{-1} \sigma) (T_{-1}^{r-2} T_{-2})^{k+1} T_{-1}^{r-2} T_{-2} \sigma \tau \\
 &= \sigma \tau^{-1} \sigma (T_{-1}^{r-2} T_{-2})^{k+1} T_{-1}^{r-2} \tau^{-1}.
 \end{aligned}$$

□

The next lemma describes what happens to the path of convergents when the continued fraction has a zero coefficient.

Lemma 2.29. *Let $[b_1, \dots, b_n]_q$ be a Rosen continued fraction with path of convergents $\langle \infty, v_1, \dots, v_n \rangle$. Then, for each integer i with $2 \leq i \leq n$, $b_i = 0$ if and only if $v_{i-2} = v_i$.*

Proof. Let $T_{b_j}(z) = b_j \lambda_q - 1/z$, so that $v_j = T_{b_1} \cdots T_{b_j}(\infty)$. Then $v_{i-2} = v_i$ if and only if $T_{b_{i-1}}^{-1}(\infty) = T_{b_i}(\infty)$; that is, if and only if $b_i \lambda_q = 0$. The result follows. □

We can now prove the first part of Theorem 2.24.

Proof of Theorem 2.24: part I. Suppose that the sequence b_2, \dots, b_n either (i) contains a 0 term; (ii) contains a subsequence of consecutive terms of the form $\pm 1^{[r]}$; or (iii) contains a subsequence of consecutive terms of the form

$$\pm(1^{[r-1]}, 2, 1^{[r-2]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-1]}).$$

We must prove that $[b_1, \dots, b_n]_q$ is not a geodesic Rosen continued fraction. Lemma 2.26 tells us that we can switch $[b_1, \dots, b_n]_q$ for

$[-b_1, \dots, -b_n]_q$ if necessary so that in cases (ii) and (iii) we have the + form of the subsequence (1s and 2s rather than -1s and -2s).

In case (i), we know from Lemma 2.29 that the path of convergents contains two equal terms, so it is not a geodesic path.

In case (ii), there is an integer $i \geq 1$ such that $b_{i+1} = \dots = b_{i+r} = 1$. Consider the alternative continued fraction

$$[b_1, \dots, b_{i-1}, b_i - 1, \underbrace{-1, -1, \dots, -1}_{r-2 \text{ copies of } -1}, b_{i+r+1} - 1, b_{i+r+2}, \dots, b_n]_q.$$

When $i = n - r$, this expression becomes

$$[b_1, \dots, b_{i-1}, b_i - 1, \underbrace{-1, -1, \dots, -1}_{r-2 \text{ copies of } -1}]_q.$$

The alternative continued fraction is shorter than $[b_1, \dots, b_n]_q$, and using Lemma 2.27 we can check that the two continued fractions have the same value:

$$\begin{aligned} T_{b_1} \dots T_{b_n}(\infty) &= T_{b_1} \dots T_{b_i} T_1^r T_{b_{i+r+1}} \dots T_{b_n}(\infty) \\ &= T_{b_1} \dots T_{b_i} \sigma \tau^{-1} \sigma T_{-1}^{r-2} \tau^{-1} T_{b_{i+r+1}} \dots T_{b_n}(\infty) \\ &= T_{b_1} \dots T_{b_{i-1}} T_{b_i-1} T_{-1}^{r-2} T_{b_{i+r+1}-1} T_{b_{i+r+2}} \dots T_{b_n}(\infty). \end{aligned}$$

Therefore $[b_1, \dots, b_n]_q$ is not a geodesic Rosen continued fraction.

In case (iii), there are integers i and j with $1 \leq i < j \leq n$ such that

$$b_{i+1}, \dots, b_j = 1^{[r-1]}, 2, 1^{[r-2]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-1]}.$$

Let $b_{i+1}^*, \dots, b_{j-2}^*$ be the shorter sequence given by

$$-b_{i+1}^*, \dots, -b_{j-2}^* = 1^{[r-2]}, 2, 1^{[r-2]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-2]}.$$

Consider the continued fraction

$$[b_1, \dots, b_{i-1}, b_i - 1, b_{i+1}^*, \dots, b_{j-2}^*, b_{j+1} - 1, b_j, \dots, b_n]_q$$

(with the obvious interpretation when $j = n$). This is shorter than $[b_1, \dots, b_n]_q$, and using Lemma 2.28 you can check that the two continued fractions have the same value. So once again $[b_1, \dots, b_n]_q$ is not a geodesic Rosen continued fraction. \square

The first part of the proof of Theorem 2.25 (which says that if b_2, \dots, b_n contains a subsequence of one of the given types then $[b_1, \dots, b_n]_q$ is not a geodesic Rosen continued fraction) follows on exactly the same lines as the proof given above. The only significant difference is that instead of the identities in Lemma 2.27 and Lemma 2.28 we need $T_1^r = \sigma\tau^{-1}\sigma T_{-1}^{r-1}\tau^{-1}$ and

$$\begin{aligned} T_1^{r-1}(T_2 T_1^{r-1} T_2 T_1^{r-2})^k T_2 T_1^{r-1} T_2 T_1^{r-1} \\ = \sigma\tau^{-1}\sigma(T_{-1}^{r-1} T_{-2} T_{-1}^{r-2} T_{-2})^{k+1} T_{-1}^{r-1} \tau^{-1}, \end{aligned}$$

where $q = 2r + 1$. To prove the latter identity, it is helpful to first observe that

$$T_2 T_1^{r-1} T_2 T_1^{r-2} = (\tau\sigma\tau^{-1}\sigma) T_{-1}^{r-2} T_{-2} T_{-1}^{r-1} T_{-2} (\tau\sigma\tau^{-1}\sigma)^{-1};$$

we omit the details.

2.4.2 PROOF OF THEOREM 2.24 AND THEOREM 2.25: PART II

In this section we prove the more difficult part of Theorem 2.24; the proof of the second part of Theorem 2.25 is similar so we omit it. Our method is thoroughly different to that of the previous section, and uses basic properties of q -chains. In fact, both parts of the two

theorems could be proved using the techniques of this section, as many of the arguments we present can, with care, be reversed.

Let us start by introducing some new terminology for paths, which involves a concept that we met earlier. A path $\langle v_0, \dots, v_n \rangle$ in \mathcal{F}_q is said to *backtrack* if $v_i = v_{i+2}$ for some integer i with $0 \leq i \leq n-2$. That is, a path backtracks if it has a subpath that proceeds from one vertex to a neighbouring vertex and then immediately back again.

We define P_1, \dots, P_m to be the q -chain from a vertex x to a non-adjacent vertex y in \mathcal{F}_q . Let a_i, b_i, μ_i , and ν_i be the vertices and paths associated to this q -chain that were introduced at the start of Section 2.3; this is illustrated in Figure 2.13. Given two paths $\gamma = \langle v_0, \dots, v_r \rangle$ and $\delta = \langle w_0, \dots, w_s \rangle$ in \mathcal{F}_q such that $v_r = w_0$, we define $\gamma\delta$ to be the path $\langle v_0, \dots, v_r, w_1, \dots, w_s \rangle$. With this notation we can distinguish two particular paths from x to y in the q -chain, namely $\alpha = \mu_1 \mu_2 \dots \mu_m$ and $\beta = \nu_1 \nu_2 \dots \nu_m$. We refer to paths in \mathcal{F}_q of this type as *outer paths*. More specifically, paths of the same type as α are called *clockwise outer paths* and paths of the same type as β are called *anticlockwise outer paths*.

The notation for the q -chain from x to y that was introduced in the previous paragraph will be retained for the rest of this section.

Lemma 2.30. *Let x and y be two non-adjacent vertices of \mathcal{F}_q . Suppose that χ is a path from x to y that lies in the q -chain from x to y and does not backtrack, and suppose that χ is not an outer path. Then χ intersects every path from x to y in a vertex other than x or y .*

Proof. As χ is not an outer path, it must traverse one of the edges $\{a_i, b_i\}$ for some integer i with $1 \leq i \leq m-1$. By Lemma 2.5, every path from x to y contains one of the vertices a_i or b_i , so χ intersects every path from x to y at one of these vertices. \square

The outer paths α and β may be equal in length, or one may be longer than the other. If one is longer than the other, then the longer

one is called a *circuitous path*. We define a *minimal circuitous path* to be a circuitous path whose non-trivial subpaths are all geodesic paths. Notice that a circuitous path $\langle v_0, \dots, v_n \rangle$ is a minimal circuitous path if and only if both $\langle v_1, \dots, v_n \rangle$ and $\langle v_0, \dots, v_{n-1} \rangle$ are geodesic paths.

Theorem 2.31. *A path in \mathcal{F}_q from x to y is a geodesic path if and only if it does not backtrack and it does not contain a minimal circuitous subpath.*

Proof. If a path backtracks or contains a circuitous subpath, then, by definition, it is not a geodesic path. To prove the converse, suppose that $\gamma = \langle v_0, \dots, v_n \rangle$ is a path in \mathcal{F}_q from x to y that is not a geodesic path. Suppose also that the path does not backtrack. We must prove that γ contains a minimal circuitous subpath. To do this, we can, by restricting to a subpath of γ if necessary, assume that every non-trivial subpath of γ is a geodesic path.

Let us first consider the cases in which x and y are either equal or adjacent. Elementary arguments show that, given the conditions just stated, the only possibility is that $\gamma = \langle x, v_1, v_2, y \rangle$, where x, v_1, v_2 , and y are distinct vertices, and x and y are adjacent. This can only happen if $q = 4$ and γ completes three sides of a face of \mathcal{F}_q , in which case γ is a minimal circuitous path.

Suppose now that x and y are neither equal nor adjacent. We will show that γ is contained within the q -chain from x to y . Theorem 2.6 shows that any path from x to y must pass through one of the y -parents of x . Let i be the smallest positive integer such that v_i is one of the y -parents of x . Since $\langle x, v_1, \dots, v_i \rangle$ is a geodesic path, it must be that $i = 1$, because otherwise $\langle x, v_i \rangle$ is a shorter path from x to v_i . Lemma 2.15 tells us that v_1 belongs to the q -chain from x to y . Repeating this argument we see that all vertices v_0, \dots, v_n belong to the q -chain from x to y .

Now let δ be a geodesic path from x to y , which, by Theorem 2.16, also lies in the q -chain from x to y . The paths γ and δ can only in-

intersect at the vertices x and y , because if they intersect at some other vertex z , then one of the subpaths of γ from x to z or from z to y is not a geodesic path. It follows from Lemma 2.30 that γ and δ are both outer paths, and since γ is not a geodesic path it must be a circuitous path. In fact γ is a minimal circuitous path, as each of its non-trivial subpaths is a geodesic path. \square

Recall the notation for the outer paths α and β defined before Lemma 2.30.

Lemma 2.32. *If α is a minimal circuitous path, then β is the only geodesic path from x to y .*

Proof. Let δ be a geodesic path from x to y . By Theorem 2.16, this path is contained in the q -chain from x to y . Suppose that it contains one of the vertices a_i , where $1 \leq i \leq m-1$. Since $\mu_1 \cdots \mu_i$ is a geodesic path from x to a_i (as it is a non-trivial subpath of α) and $\mu_{i+1} \cdots \mu_m$ is a geodesic path from a_i to y , it follows that α is a geodesic path, which is a contradiction. Therefore δ does not contain any of the vertices a_i , so it must equal β . \square

We can characterise the minimal circuitous paths precisely, and we do so in Lemma 2.33 and Theorem 2.34, below.

Lemma 2.33. *Let q be equal to either $2r$ or $2r+1$, where $r \geq 2$. If α is a minimal circuitous path on a q -chain of length 1 (that is, $m=1$), then $|\mu_1| = r+1$.*

Proof. Since α is a circuitous path, and $m=1$, we have $|\mu_1| > |v_1|$. Since $|\mu_1| + |v_1| \geq 2r$, we see that $|\mu_1| \geq r+1$. If $|\mu_1| > r+1$, then α contains a non-geodesic subpath (just remove the final vertex from μ_1). Therefore $|\mu_1| = r+1$. \square

Theorem 2.34. Suppose that $q \geq 4$. Let α be a clockwise outer path on a q -chain of length at least 2 (that is, $m \geq 2$). If α is a minimal circuitous path, then

$$|\mu_1|, \dots, |\mu_m| = \begin{cases} \underbrace{r, r-1, r-1, \dots, r-1, r}_{m \geq 2} & \text{if } q = 2r, \\ \underbrace{r, r, r-1, r, r-1, \dots, r-1, r, r}_{m \geq 3} & \text{if } q = 2r+1. \end{cases} \quad (2.1)$$

The sequences $|\mu_1|, \dots, |\mu_m|$ described above are

$$r, r, \quad r, r-1, r, \quad r, r-1, r-1, r,$$

and so on, when $q = 2r$, and

$$r, r, r \quad r, r, r-1, r, r \quad r, r, r-1, r, r-1, r, r,$$

and so on, when $q = 2r+1$. There is a converse to Theorem 2.34, which says that if α is an outer path and $|\mu_1|, \dots, |\mu_m|$ is one of these sequences, then α is a minimal circuitous path. We do not prove this converse result as we do not need it.

The following two lemmas are needed to prove Theorem 2.34.

Lemma 2.35. Let q be equal to either $2r$ or $2r+1$, where $r \geq 2$. If α is a minimal circuitous path on a q -chain of length at least 2 (that is, $m \geq 2$), then $r-1 \leq |\mu_i| \leq r$ for $i = 1, \dots, m$. Furthermore, $|\mu_1| = |\mu_m| = r$.

Proof. Suppose that $|\mu_i| \geq r+1$. Then μ_i is not a geodesic path, which is impossible, as it is a subpath of α . Therefore $|\mu_i| \leq r$.

Suppose next that $|\mu_i| \leq r-2$, where $1 < i < m$. Let δ be the path from x to y given by

$$\delta = v_1 \cdots v_{i-1} \langle b_{i-1}, a_{i-1} \rangle \mu_i \langle a_i, b_i \rangle v_{i+1} \cdots v_m.$$

Since $|\mu_i| + |\nu_i| + 2 \geq 2r$, we see that

$$|\beta| - |\delta| = |\nu_i| - |\mu_i| - 2 \geq 2(r - 2 - |\mu_i|) \geq 0.$$

Therefore $|\delta| \leq |\beta|$, so δ is a geodesic path from x to y , which contradicts Lemma 2.32. Therefore $|\mu_i| \geq r - 1$ when $1 < i < m$.

Finally, we prove that $|\mu_1| = r$; the proof that $|\mu_m| = r$ is similar and is omitted. Suppose that $|\mu_1| \leq r - 1$. Let δ be the path from x to y given by $\delta = \mu_1 \langle a_1, b_1 \rangle \nu_2 \cdots \nu_m$. Since $|\mu_1| + |\nu_1| + 1 \geq 2r$, we see that $|\beta| - |\delta| = |\nu_1| - |\mu_1| - 1 \geq 2(r - 1 - |\mu_i|) \geq 0$. Therefore δ is a geodesic path, which contradicts Lemma 2.32. Therefore $|\mu_1| \geq r$. Since we have already proved that $|\mu_1| \leq r$, we conclude that $|\mu_1| = r$. \square

Lemma 2.36. *Let $q = 2r + 1$, where $r \geq 2$. If α is a minimal circuitous path on a q -chain of length at least 3 (that is, $m \geq 3$), then $|\mu_2| = |\mu_{m-2}| = r$. Also, for no integer i with $1 < i < m - 1$ are $|\mu_i|$ and $|\mu_{i+1}|$ both equal to $r - 1$.*

Proof. Let us first prove that $|\mu_2| = r$. The proof that $|\mu_{m-2}| = r$ is similar and omitted. We know from Lemma 2.35 that $|\mu_2|$ is either $r - 1$ or r . If $|\mu_2| = r - 1$, then the path

$$\delta = \mu_1 \mu_2 \langle a_2, b_2 \rangle \nu_2 \cdots \nu_m$$

is a geodesic path from x to y , which contradicts Lemma 2.32. Therefore $|\mu_2| = r$.

For the second assertion of the lemma, suppose that $|\mu_i| = |\mu_{i+1}| = r - 1$ for some integer i , where $1 < i < m$. In this case, the path

$$\delta = \nu_1 \cdots \nu_{i-1} \langle b_{i-1}, a_{i-1} \rangle \mu_i \mu_{i+1} \langle a_{i+1}, b_{i+1} \rangle \nu_{i+2} \cdots \nu_m$$

is a geodesic path from x to y , which contradicts Lemma 2.32. \square

Proof of Theorem 2.34. Suppose first that α is an outer path and $|\mu_1|, \dots, |\mu_m|$ takes one of the values given in Equation (2.1); we do not yet assume that α is a minimal circuitous path. Let us prove that α is not a geodesic path. To do this, we will show that $|\beta| < |\alpha|$. This is true when $q = 2r$, because $|\alpha| = m(r-1) + 2$ and $|\alpha| + |\beta| = 2m(r-1) + 2$, so $|\beta| = m(r-1)$. It is also true when $q = 2r+1$, because $|\alpha| = (2r-1)m/2 + 3/2$ and $|\alpha| + |\beta| = (2r-1)m + 2$, so $|\beta| = (2r-1)m/2 + 1/2$.

Now let us assume that α is a minimal circuitous path. By Lemma 2.35, $|\mu_i|$ is either $r-1$ or r for $i = 1, \dots, m$.

Suppose that $q = 2r$. Lemma 2.35 tells us that $|\mu_1| = |\mu_m| = r$. Suppose that $|\mu_i| = r$ for some integer i with $1 < i < m$; in fact, let i be the smallest such integer. Then $\mu_1 \cdots \mu_i$ is not a geodesic path, as we demonstrated at the start of this proof, which is a contradiction, as it is a subpath of $\mu_1 \cdots \mu_m$. Hence $|\mu_i| = r-1$ for $1 < i < m$, as required.

Suppose now that $q = 2r+1$. Lemmas 2.35 and 2.36 tell us that $|\mu_1| = |\mu_2| = |\mu_{m-1}| = |\mu_m| = r$. The path $\mu_1 \cdots \mu_m$ is not circuitous when m is 2 or 4 (because it is the same length as β in each case). However, it is circuitous when $m = 3$.

Let us assume then that $m \geq 5$. We know that $|\mu_3| = r-1$, because if $|\mu_3| = r$, then the subpath $\mu_1 \mu_2 \mu_3$ of α is not a geodesic path. The second assertion of Lemma 2.36 tells us that $|\mu_4| = r$. Next, if $|\mu_5| = r$, then, as we saw at the start of this proof, $\mu_1 \cdots \mu_5$ is not a geodesic path, which can only be so if $m = 5$. Therefore $|\mu_5| = r-1$ when $m > 5$. Arguing repeatedly in this fashion, we see that $|\mu_1|, \dots, |\mu_m| = r, r, r-1, r, r-1, \dots, r-1, r, r$, where m is odd and at least 3. \square

The next lemma allows us to move from paths to continued fractions.

Lemma 2.37. *Let $\langle v_0, v_1, \dots, v_n \rangle$, where $v_0 = \infty$, be the path of convergents of a Rosen continued fraction $[b_1, \dots, b_n]_q$. Suppose that $\langle v_k, \dots, v_l \rangle$, where $0 \leq k < l \leq n$ and $k+2 \leq l$, is a clockwise outer path $\alpha = \mu_1 \cdots \mu_m$ such that $|\mu_i| \geq 1$ for $i = 1, \dots, m$. Then*

$$b_{k+2}, \dots, b_l = 1^{[|\mu_1|-1]}, 2, 1^{[|\mu_2|-1]}, 2, \dots, 2, 1^{[|\mu_m|-1]}.$$

Proof. Recall the function ϕ defined near the end of Section 2.1.3. Lemma 2.3 tells us that $\phi(v_{i-1}, v_i, v_{i+1}) = b_{i+1}$ for $i = k+1, \dots, l-1$, so we need only calculate these values of ϕ one by one. If v_i is not one of the vertices a_j , then v_{i-1} , v_i , and v_{i+1} lie in that order clockwise round a face of \mathcal{F}_q . Therefore $\phi(v_{i-1}, v_i, v_{i+1}) = 1$. If v_i is one of the vertices a_j , then because $|\mu_i|$ and $|\mu_{i+1}|$ are both at least 1, v_{i-1} , v_i , and v_{i+1} lie in that order clockwise around a $(2q-2)$ -gon comprised of two adjacent faces of \mathcal{F}_q that meet along an edge that has v_i as a vertex. If we map v_i to ∞ by an element of Γ_q , then we see that $\phi(v_{i-1}, v_i, v_{i+1}) = 2$. Therefore b_{k+2}, \dots, b_l is of the given form. \square

Finally, we are able to prove the second part of Theorem 2.24.

Proof of Theorem 2.24: part II. Suppose that $[b_1, \dots, b_n]_q$ is not a geodesic Rosen continued fraction. We must prove that the sequence b_2, \dots, b_n either (i) contains a 0 term; (ii) contains a subsequence of consecutive terms of the form $\pm 1^{[r]}$; or (iii) contains a subsequence of consecutive terms of the form

$$\pm(1^{[r-1]}, 2, 1^{[r-2]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-1]}).$$

Since the path of convergents of $\gamma = \langle \infty, v_1, \dots, v_n \rangle$ is not a geodesic path, Theorem 2.31 tells us that it either backtracks or contains a minimal circuitous path. If it backtracks, then, by Lemma 2.29, one of the coefficients b_2, \dots, b_n is 0, which is case (i). Otherwise, γ con-

tains a minimal circuitous subpath. Suppose for the moment that this minimal circuitous subpath is a *clockwise* outer path, namely $\alpha = \mu_1 \cdots \mu_m$. Lemma 2.33 and Theorem 2.34 tell us that either $m = 1$ and $|\mu_1| = r + 1$ or $m > 1$ and

$$|\mu_1|, \dots, |\mu_m| = r, r-1, r-1, \dots, r-1, r.$$

It follows from Lemma 2.37 that there are integers $0 \leq k < l \leq n$ with $k+2 \leq l$ such that the sequence b_{k+2}, \dots, b_l is $1^{[r]}$, when $m = 1$, or

$$1^{[r-1]}, 2, 1^{[r-2]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-1]},$$

when $m > 1$. These are cases (ii) and (iii).

Earlier we assumed that γ contained a minimal circuitous subpath that was a *clockwise* outer path; suppose now that it is an *anticlockwise* outer path β . Recall that the map $\kappa(z) = -\bar{z}$ is an anticonformal transformation of the upper half-plane that induces an automorphism of \mathcal{F}_q . As we saw in Lemma 2.26, the path of convergents of the continued fraction $[-b_1, \dots, -b_n]_q$ is $\kappa(\gamma)$. The path $\kappa(\beta)$ is a minimal circuitous subpath of $\kappa(\gamma)$, but it is a clockwise outer path rather than an anticlockwise outer path, as κ reverses the orientation of cycles in \mathcal{F}_q . Since, by Lemma 2.26, $[-b_1, \dots, -b_n]_q$ is not a geodesic Rosen continued fraction, the argument from the previous paragraph shows that $-b_2, \dots, -b_n$ contains a subsequence of consecutive terms of the form $1^{[r]}$ or $1^{[r-1]}, 2, 1^{[r-2]}, \dots, 1^{[r-2]}, 2, 1^{[r-1]}$. Therefore we see once again that one of statements (ii) or (iii) holds. \square

The proof of the second part of Theorem 2.25 mirrors the proof given above almost exactly, but with the sequence of 1s and 2s and the sequence of rs and $(r-1)$ s suitably modified.

2.5 INFINITE ROSEN CONTINUED FRACTIONS

So far in this chapter, for simplicity, we have focused on finite continued fractions; however, most of our theorems and techniques generalise in a straightforward fashion to infinite continued fractions. Here we briefly discuss the theory of infinite Rosen continued fractions. We give a remarkably mild sufficient condition for an infinite Rosen continued fraction to converge, introduce the notion of geodesic infinite Rosen continued fractions, and prove the following theorem, which is an analogue for infinite Rosen continued fractions of Theorem 2.13.

Theorem 2.38. *For each $q \geq 3$, the nearest-integer algorithm applied to a real number y that is not a vertex of \mathcal{F}_q gives rise to a geodesic Rosen continued fraction expansion of y .*

Recall that an *infinite path* in \mathcal{F}_q is a sequence of vertices v_0, v_1, \dots such that $v_{i-1} \sim v_i$ for $i = 1, 2, \dots$. The convergents of an infinite Rosen continued fraction form an infinite path $\langle \infty, v_1, v_2, \dots \rangle$ in \mathcal{F}_q , and conversely each infinite path of this type is comprised of the convergents of a unique infinite Rosen continued fraction. The proof of this fact is much the same as the proof of Theorem 2.2 for finite Rosen continued fractions.

2.5.1 CONVERGENCE OF INFINITE ROSEN CONTINUED FRACTIONS

We wish to prove the following sufficient condition for the convergence of an infinite Rosen continued fraction.

Theorem 2.39. *If the sequence of convergents of an infinite Rosen continued fraction does not contain infinitely many terms that are equal, then the continued fraction converges.*

Note that there is an obvious converse to Theorem 2.39: if an infinite Rosen continued fraction converges to some value x , then it can only have infinitely many convergents equal to some vertex y if $x = y$. For example, $0, 1, 0, 1/2, 0, 1/3, \dots$ is the sequence of convergents of the integer continued fraction $[0, -1, 0, -2, 0, -3, \dots]_3$, and this sequence converges to the vertex 0 of \mathcal{F}_3 .

To prove Theorem 2.39, we use the following lemma.

Lemma 2.40. *Suppose that x and y are distinct elements of $\hat{\mathbb{R}}$ that are not adjacent vertices of \mathcal{F}_q . Then there are two vertices u and v of some face of \mathcal{F}_q such that x and y lie in distinct components of $\hat{\mathbb{R}} \setminus \{u, v\}$.*

Proof. If x and y are both vertices of \mathcal{F}_q , then this assertion follows from Lemma 2.4. If one of x and y is a vertex of \mathcal{F}_q (say x) and the other is not, then after applying an element of Γ_q we may assume that $x = \infty$, in which case we can choose u and v to be the integer multiples of λ_q that lie either side of y on the real line. Suppose finally that neither x nor y are vertices of \mathcal{F}_q . The hyperbolic line from x to y must intersect an edge of \mathcal{F}_q (else this hyperbolic line disconnects \mathcal{F}_q) and the endpoints of this edge are the required vertices u and v . \square

Proof of Theorem 2.39. We prove the contrapositive of Theorem 2.39. Suppose that the sequence of convergents v_1, v_2, \dots of an infinite Rosen continued fraction diverges. Then this sequence has two distinct accumulation points, x and y . Assume for the moment that x and y are not adjacent vertices of \mathcal{F}_q . Then Lemma 2.40 tells us that there are two vertices u and v of some face of \mathcal{F}_q such that x and y lie in distinct components of $\hat{\mathbb{R}} \setminus \{u, v\}$. Since x and y are both accumulation points of the sequence v_1, v_2, \dots , the path of convergents $\langle \infty, v_1, v_2, \dots \rangle$ contains infinitely many subpaths that pass from one component of $\hat{\mathbb{R}} \setminus \{u, v\}$ into the other. It follows from Lemma 2.5 that the sequence v_1, v_2, \dots either contains infinitely many terms equal to u or else it contains infinitely many terms equal to v .

Suppose now that x and y are adjacent vertices of \mathcal{F}_q , which, after applying an element of Γ_q , we can assume are 0 and ∞ . Since the sequence of convergents accumulates at 0 it contains infinitely many terms inside one of the intervals $[-\lambda_q, 0]$ or $[0, \lambda_q]$, and it also contains infinitely many terms that lie outside the union of these two intervals. It follows from Lemma 2.5 that the sequence contains infinitely many equal terms, all equal to one of $-\lambda_q$, 0, or λ_q . \square

2.5.2 INFINITE GEODESIC ROSEN CONTINUED FRACTIONS

Let us now discuss infinite geodesic Rosen continued fractions. An infinite Rosen continued fraction $[b_1, b_2, \dots]_q$ is a *geodesic* Rosen continued fraction if $[b_1, \dots, b_k]_q$ is a geodesic Rosen continued fraction for all $k = 1, 2, \dots$. An infinite path $\langle v_0, v_1, \dots \rangle$ in \mathcal{F}_q is said to be a *geodesic path* if $\langle v_0, v_1, \dots, v_n \rangle$ is a geodesic path for each positive integer n . It follows that $[b_1, b_2, \dots]_q$ is a geodesic Rosen continued fraction if and only if its path of convergents is a geodesic path. We can determine whether $[b_1, b_2, \dots]_q$ is a geodesic Rosen continued fraction using Theorem 2.24 and Theorem 2.25.

The next observation is an immediate corollary of Theorem 2.39.

Corollary 2.41. *Every infinite geodesic Rosen continued fraction converges.*

Earlier we proved that, given vertices x and y of \mathcal{F}_q , we can construct a geodesic path from x to y by iterating the map α_y . In fact, we can define α_y even when y is not a vertex of \mathcal{F}_q . In this case, for any vertex x , the iterates $x, \alpha_y(x), \alpha_y^2(x), \dots$ form an infinite path in \mathcal{F}_q . We can see that this is a geodesic path because, for any positive integer n , we can choose a vertex y_0 of \mathcal{F}_q that is sufficiently close to y in the spherical metric on $\hat{\mathbb{R}}$ that the two paths $\langle x, \alpha_y(x), \dots, \alpha_y^n(x) \rangle$ and $\langle x, \alpha_{y_0}(x), \dots, \alpha_{y_0}^n(x) \rangle$ are identical, and we know that the latter is a geodesic path. Corollary 2.41 now tells us that the sequence

$x, \alpha_y(x), \alpha_y^2(x), \dots$ converges in $\hat{\mathbb{R}}$, and a short argument that we omit shows that the limit must be y . This gives us the following theorem.

Theorem 2.42. *Given a vertex x of \mathcal{F}_q and a real number y that is not a vertex of \mathcal{F}_q , the infinite path $\langle x, \alpha_y(x), \alpha_y^2(x) \dots \rangle$ is a geodesic path in \mathcal{F}_q which converges in $\hat{\mathbb{R}}$ to y .*

Theorem 2.43. *For each $q \geq 3$, the nearest-integer algorithm applied to a vertex y of \mathcal{F}_q gives rise to a geodesic Rosen continued fraction expansion of y .*

We can now prove Theorem 2.38.

Proof of Theorem 2.38. The path $\langle \infty, \alpha_\infty(y), \alpha_\infty^2(y) \dots \rangle$ is the path of convergents of the Rosen continued fraction expansion of y obtained by applying the nearest-integer algorithm. We have seen this already when y is a vertex of \mathcal{F}_q , and the same is true when y is not a vertex. Theorem 2.42 says that this path is a geodesic path, so we have proved Theorem 2.13 for infinite Rosen continued fractions. □

We finish here with an example to show that a real number may have infinitely many infinite geodesic Rosen continued fraction expansions. In our example $q = 4$, but there are similar examples for other values of q . The simplest way to describe the example is using the infinite q -chain suggested by Figure 2.26. The number y is equal to $[2, 2, \dots]_4$, and we can see from the infinite q -chain that there are infinitely many geodesic paths from ∞ to y .

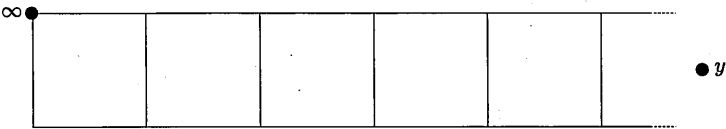


Figure 2.26: There are infinitely many geodesic paths from ∞ to y .

EVEN-INTEGER CONTINUED FRACTIONS

Closely related to Rosen continued fractions are even-integer continued fractions. Unlike with Rosen continued fractions, however, there has been little research into the properties of even-integer continued fractions. Although the study of even-integer continued fractions began with the work of Fritz Schweiger [49, 50] in the 1980s, the first comprehensive discussion of even-integer continued fractions did not appear until 1996, with the work of Cornelis Kraaikamp and Artur Lopes [29]. Similar results to those found in [29] are obtained in the recent paper [43], but there are many properties of even-integer continued fractions that remain unexplored.

In this chapter we describe how even-integer continued fractions can be represented by simple paths in a graph embedded in \mathbb{H} that is a natural analogue of the Farey graph introduced in Section 1.2.2. This graph is a tree, and a consequence of this is that every even-integer continued fraction is geodesic. We turn our attention away from the notion of geodesic continued fractions, and instead expound the number-theoretic properties of even-integer continued fractions. We use our geometric perspective to develop a theory of even-integer continued fractions that is remarkably similar to the theory of regular continued fractions that can be found in, for example, [19, 23], and give insight into the work of Kraaikamp and Lopes in [29].

The structure of this chapter is as follows. In Section 3.1 we define even-integer continued fractions and show how they can be viewed as simple paths in a tree embedded in \mathbb{H} . In Section 3.2 we look at the representation of real numbers by even-integer continued frac-

tions, and we show that an even-integer continued fraction expansion of a real number can be obtained using an analogue of the familiar nearest-integer algorithm. We prove analogues of classical theorems of Serret and Lagrange. We also discuss how the even-integer continued fraction expansion of a real number can be obtained from its regular continued fraction expansion. In Section 3.3 we look at Diophantine approximation, and show that even-integer continued fractions have properties that are analogous to the properties of regular continued fractions. We determine the Hurwitz constant for this approximation. Finally, in Section 3.4 we show that both Rosen continued fractions and even-integer continued fractions are a part of a wider class of continued fractions, which we call λ -continued fractions, and discuss how the techniques used to study Rosen and even-integer continued fractions might be extended to study λ -continued fractions.

3.1 EVEN-INTEGER CONTINUED FRACTIONS AND HYPERBOLIC GEOMETRY

An *even-integer continued fraction* is a continued fraction of the form

$$b_1 + \frac{-1}{b_2 + \frac{-1}{b_3 + \dots}},$$

where each b_i is an even integer, and $b_i \neq 0$ for $i > 1$. Using the notation introduced in Section 1.1.1, we denote an infinite even-integer continued fraction by $[b_1, b_2, \dots]$, and a finite even-integer continued fraction by $[b_1, \dots, b_n]$.

In this section we will show that even-integer continued fractions can be viewed as simple paths in a certain graph embedded in \mathbb{H} .

This graph arises naturally from the action of the theta group on \mathbb{H} ; we begin by discussing the theta group and its relation to even-integer continued fractions.

3.1.1 THE THETA GROUP

Recall the Hecke groups $\Gamma(\lambda)$ defined in Section 2.1.1; they are the groups of Möbius transformations generated by

$$\tau_\lambda(z) = z + \lambda \quad \text{and} \quad \sigma(z) = -\frac{1}{z},$$

where either $\lambda = \lambda_q = 2\cos(\pi/q)$ or $\lambda \geq 2$. In Chapter 2 we studied the Hecke groups $\Gamma_q = \Gamma(\lambda_q)$, which give rise to Rosen continued fractions. In this chapter we discuss the Hecke group $\Gamma(\lambda)$ when $\lambda = 2$, which we denote by Γ_∞ since $\lambda_q \rightarrow 2$ as $q \rightarrow \infty$ (and to avoid confusion with the congruence subgroup $\Gamma(2)$ of the modular group Γ). The group Γ_∞ is called the *theta group*, and is important in the study of modular forms.

The group Γ_∞ is a Fuchsian group of the first kind; indeed, it is a subgroup of the modular group Γ of index three. It consists of those Möbius transformations

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c and d are integers and $ad - bc = 1$, such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2}.$$

(see, for example, [28, Corollary 4]). The generators τ_2 , which we will denote by τ from now on, and σ , satisfy the relation $\sigma^2 = I$ where I

is the identity transformation, and all other relations in τ and σ are consequences of this. It follows that Γ_∞ is isomorphic as a group to the free product $C_2 * C_\infty$.

Consider a finite even-integer continued fraction $[b_1, \dots, b_n]$. As in Section 1.2, we define a sequence of Möbius transformations

$$f_i(z) = b_i - \frac{1}{z},$$

for $i = 1, 2, \dots, n$, that is, $f_i = \tau^{b_i/2}\sigma$ using the generators τ and σ of Γ_∞ . Then the convergents of $[b_1, \dots, b_n]$ are given by

$$v_k = f_1 \cdots f_k(\infty),$$

for $k = 1, \dots, n$, and an even-integer continued fraction can be viewed as a sequence $F_k = f_1 \cdots f_k$ of Möbius transformations belonging to the theta group. Similarly, an infinite even-integer continued fraction can be viewed as an infinite sequence of Möbius transformations belonging to the theta group. The action of the theta group on \mathbb{H} gives rise to a geometric representation of even-integer continued fractions.

3.1.2 THE FAREY GRAPH \mathcal{F}_∞

The action of the theta group Γ_∞ on \mathbb{H} gives rise to a graph in which paths represent even-integer continued fractions. Before defining these graphs, we first discuss some geometric properties of the theta group. It is more convenient to work with an alternative pair of generators of Γ_∞ , namely τ and ρ where

$$\rho(z) = \tau\sigma(z) = 2 - \frac{1}{z}.$$

In \mathbb{H} , the hyperbolic quadrilateral D with vertex i and ideal vertices $1, 2 + i$ and ∞ is a fundamental domain for Γ_∞ with side-pairing transformations τ and ρ ; see [12, 22]. The quadrilateral D is shown in Figure 3.1 (a).

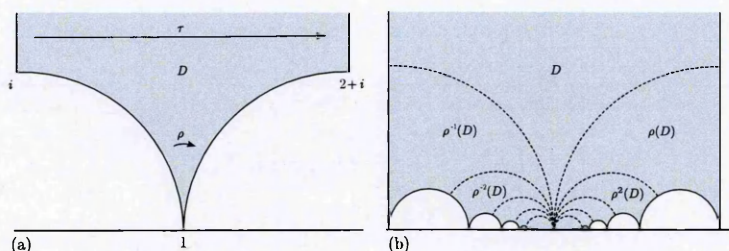
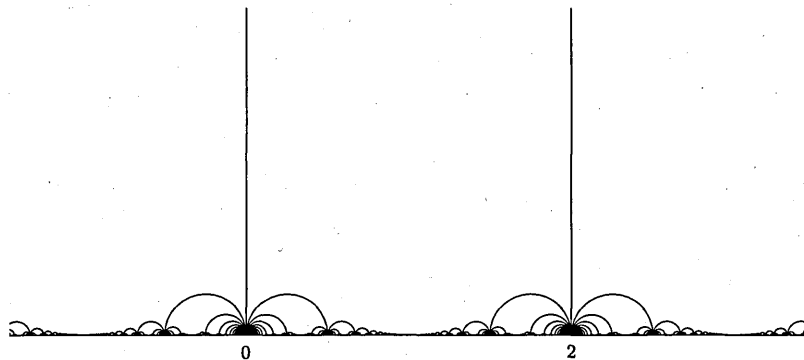


Figure 3.1: (a) A fundamental domain for Γ_∞ (b) A fundamental domain for Θ_∞

Let Θ_∞ denote the group generated by the involutions $\rho^i \sigma \rho^{-i}$ for $i \in \mathbb{Z}$, which is an infinite index normal subgroup of Γ_∞ . A fundamental domain E for Θ_∞ is given by $E = \bigcup_{i \in \mathbb{Z}} \rho^i(D)$, as shown in Figure 3.1 (b). To see that E is indeed a fundamental domain for Θ_∞ , notice that the maps $\rho^i \sigma \rho^{-i}$ are side-pairing transformations for E ; Poincaré's Theorem may then be applied. We view this fundamental domain as an ideal hyperbolic polygon with infinitely many sides, and its images under Θ_∞ tessellate \mathbb{H} . The skeleton of this tessellation is a connected plane graph, which we call a *Farey graph* and denote by \mathcal{F}_∞ . The vertices of \mathcal{F}_∞ are the ideal vertices of the tessellation; they form a countable, dense subset of the ideal boundary $\hat{\mathbb{R}}$ of \mathbb{H} . The edges of \mathcal{F}_∞ are the sides of the ideal infinitely-sided polygons in the tessellation, and the faces of \mathcal{F}_∞ are the polygons themselves. Part of \mathcal{F}_∞ is shown in Figure 3.2.

The graph \mathcal{F}_∞ is the graph introduced by Singerman in [56] as a tessellation of the hyperbolic plane that can be used as a universal cover for any map on a surface, although the definition in [56] is slightly different. The graph \mathcal{F}_∞ is also similar to one that is used in [43] to study a class of continued fractions that are related to, but not

Figure 3.2: The Farey graph \mathcal{F}_∞

exactly the same as, even-integer continued fractions. The techniques used in [43] are quite different to those used in this thesis: they are more traditional, and do not make use of hyperbolic geometry.

To determine the vertices of \mathcal{F}_∞ more precisely we use an alternative definition of \mathcal{F}_∞ to that given above. Let δ be the hyperbolic geodesic in \mathbb{H} between 0 and ∞ . Under iterates of the map ρ , this hyperbolic line is mapped to each of the sides of the fundamental domain E of Θ_∞ that is shown in Figure 3.1. It follows that \mathcal{F}_∞ is the orbit of δ under Γ_∞ , and could have been defined in this way. The transformation $\sigma(z) = -1/z$ maps ∞ to 0, so the set of vertices of \mathcal{F}_∞ is the orbit of ∞ under Γ_∞ . We therefore call the vertices of \mathcal{F}_∞ *∞ -rationals*. They consist of the set of reduced rationals whose numerator and denominator differ in parity, together with the point ∞ , which we identify with $1/0$. The ∞ -rationals are the fixed points of one of the two conjugacy classes of parabolic elements in Γ_∞ . The remaining rationals are called *1-rationals*, because they consist of the images of 1 under Γ_∞ . They are the reduced rationals whose numerator and denominator are both odd, and are the fixed points of the other of the two conjugacy classes of parabolic elements of Γ_∞ . They are the accumulation points of the vertices surrounding the faces of \mathcal{F}_∞ , and are called *face-centre points* in [56].

As in Section 2.1.2, we can also use this alternative description of the Farey graph \mathcal{F}_∞ to show that elements of Γ_∞ are automorphisms of \mathcal{F}_∞ . The map κ given by $\kappa(z) = -\bar{z}$ is an automorphism of \mathcal{F}_∞ , and hence the group $\tilde{\Gamma}_\infty$ generated by Γ_∞ and the κ , which we call the *extended theta group*, is a group of automorphisms of \mathcal{F}_∞ . These automorphisms all preserve incidence between vertices, edges and faces of \mathcal{F}_∞ . The group $\tilde{\Gamma}_\infty$ is not, however, the full group of automorphisms of \mathcal{F}_∞ . To see this, consider a map f that acts as τ on the real interval $(-1, 1)$, as τ^{-1} on the real interval $(1, 3)$ and the identity elsewhere. Conjugating f by elements of $\tilde{\Gamma}_\infty$ provides an infinite family of automorphisms of \mathcal{F}_∞ that do not lie in $\tilde{\Gamma}_\infty$.

There is a third equivalent definition of \mathcal{F}_∞ that is helpful to bear in mind. Recall from Section 1.2.2 that the Farey graph \mathcal{F} is the orbit of δ , the hyperbolic geodesic in \mathbb{H} between 0 and ∞ , under the modular group Γ . Since the theta group is a subgroup of the modular group, it follows that the Farey graph \mathcal{F}_∞ is a subgraph of the Farey graph \mathcal{F} . In fact, it is the subgraph of \mathcal{F} obtained by removing those vertices that are 1-rationals. To see this, notice that in both \mathcal{F} and \mathcal{F}_∞ two ∞ -rationals a/b and c/d are joined by an edge if and only if $|ad - bc| = 1$. A part of \mathcal{F}_∞ as a subgraph of \mathcal{F} is shown in Figure 3.3.

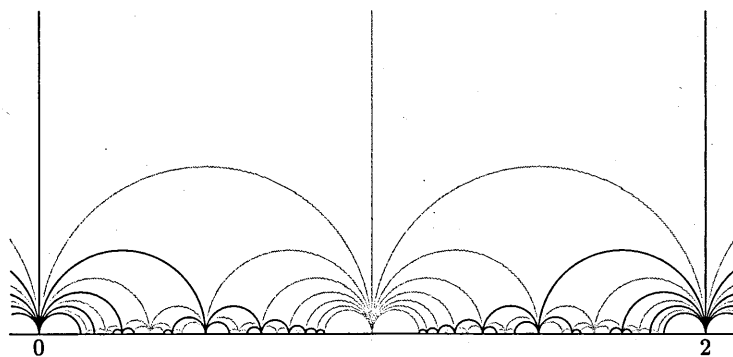


Figure 3.3: The Farey graph \mathcal{F}_∞ (black lines) and the Farey graph \mathcal{F} (both grey and black lines)

We can immediately deduce that since the neighbours of ∞ in \mathcal{F} are the integers, the neighbours of ∞ in \mathcal{F}_∞ are precisely the even integers. We can also deduce that since no two edges of \mathcal{F} may intersect one another, no two edges of \mathcal{F}_∞ may intersect either. In fact, no edge of \mathcal{F}_∞ may intersect any other edge of \mathcal{F} . Furthermore, we can now prove the following important lemma.

Lemma 3.1. *Let u be a vertex of \mathcal{F}_∞ lying in the interval $(-1, 1)$, and let v be a vertex of \mathcal{F}_∞ lying outside this interval. Then any path in \mathcal{F}_∞ from u to v passes through the vertex ∞ .*

Proof. No edge of \mathcal{F}_∞ may intersect the edge in \mathcal{F} between 1 and ∞ , nor may it intersect the edge in \mathcal{F} between -1 and ∞ . Since the points 1 and -1 are not vertices of \mathcal{F}_∞ , it follows that the only edge of \mathcal{F}_∞ that has one endpoint within $(-1, 1)$ and the other endpoint outside of $(-1, 1)$ is the edge between 0 and ∞ . It follows that any path between a vertex in $(-1, 1)$ and a vertex outside of $(-1, 1)$ must pass along this edge, and hence pass through ∞ . \square

The following lemma about the structure of \mathcal{F} allows us to prove that \mathcal{F}_∞ is a tree.

Lemma 3.2. *Any face of \mathcal{F} has precisely two vertices that are ∞ -rationals, and one vertex that is a 1-rational.*

Proof. We show first that no two adjacent vertices of \mathcal{F} are 1-rationals. It then follows that no face of \mathcal{F} may have more than one vertex that is a 1-rational. Suppose that $u = a/b$ and $v = c/d$ are two 1-rationals. Then a , b , c and d are all odd. But then $|ad - bc| \neq 1$, which contradicts the fact that u and w are neighbours in \mathcal{F} . It remains to show that no face of \mathcal{F} may have three ∞ -rational vertices. Let u , v and w

be three vertices of a face of \mathcal{F} . It is well known (see, for example, [19, Theorem 29]) that by relabelling u , v and w if necessary, we can write

$$u = \frac{a}{b}, \quad v = \frac{c}{d}, \quad \text{and} \quad w = \frac{a+c}{b+d}.$$

Therefore if both u and v are ∞ -rational, then w is not. \square

Theorem 3.3. *The graph \mathcal{F}_∞ is a tree.*

Proof. We begin by proving that \mathcal{F}_∞ is connected. Let u and v be two vertices of \mathcal{F}_∞ , and suppose that u and v are not joined by a path in \mathcal{F}_∞ . Since u and v are vertices of \mathcal{F} , we can consider the 3-chain from u to v , as defined in Section 2.3.1. Since u and v are not joined by a path in \mathcal{F}_∞ , the subgraph of the 3-chain obtained by restricting to those vertices lying in \mathcal{F}_∞ is not connected. It follows that there is some face in the 3-chain from u to v such that at least two of the vertices of this face are 1-rational. But this contradicts Lemma 3.2. Therefore, x and y must be joined by a path in \mathcal{F}_∞ , and \mathcal{F}_∞ is connected.

It remains to show that \mathcal{F}_∞ contains no closed simple path. Suppose, on the contrary, that γ is a closed simple path in \mathcal{F}_∞ with vertices v_0, v_1, \dots, v_n . We may assume, after applying an element of Γ_∞ , that $v_0 = \infty$ and $v_1 = 0$. Since v_{n-1} is a neighbour of v_0 , and $v_{n-1} \neq v_1$, v_{n-1} does not lie in the real interval $(-1, 1)$. Then, since $v_i \neq v_0$ for $i \neq 0$, there is a path from v_1 to v_{n-1} in \mathcal{F}_∞ that does not pass through ∞ . But this contradicts Lemma 3.1. Therefore, \mathcal{F}_∞ contains no closed simple paths.

This completes the proof of the theorem. \square

3.1.3 PATHS IN \mathcal{F}_∞

We are now in a position to state the theorem that explains the correspondence between even-integer continued fractions and paths in

\mathcal{F}_∞ . For now, we will concentrate on finite even-integer continued fractions and finite paths.

Theorem 3.4. *Let x be a vertex of \mathcal{F}_∞ other than ∞ . Then the vertices v_1, \dots, v_n of \mathcal{F}_∞ , with $v_n = x$, are the consecutive convergents of some even-integer continued fraction expansion of x if and only if $\langle \infty, v_1, \dots, v_n \rangle$ is a simple path in \mathcal{F}_∞ from ∞ to x .*

Proof. Suppose first that v_1, \dots, v_n are the consecutive convergents of the even-integer continued fraction expansion $[b_1, \dots, b_n]$ of x . We can prove that $\langle \infty, v_1, \dots, v_n \rangle$ is a path in \mathcal{F}_∞ from ∞ to x using the same proof as that of Theorem 2.2; we omit the details here. If $\langle \infty, v_1, \dots, v_n \rangle$ is not simple, then there is some $i = 2, \dots, n$ for which $v_{i-2} = v_i$. But then $f_{i-1}^{-1}(\infty) = f_i(\infty)$, and $b_i = 0$. This contradicts the definition of an even-integer continued fraction, and so the path $\langle \infty, v_1, \dots, v_n \rangle$ must be simple.

Conversely, suppose that $\langle \infty, v_1, \dots, v_n \rangle$ is a simple path from ∞ to x . We can prove that there is an even-integer continued fraction $[b_1, \dots, b_n]$ with convergents v_1, \dots, v_n using the same proof as that of Theorem 2.2; again, we omit the details here. \square

As in Section 1.2 and Section 2.1, we call $\langle \infty, v_1, \dots, v_n \rangle$ the *path of convergents* of $[b_1, \dots, b_n]$. This theorem is very similar to [43, Theorem 3.1], although the proof in [43] does not use hyperbolic geometry.

There is a simple way to move between an even-integer continued fraction and its path of convergents, which is similar to the way of moving between a Rosen continued fraction and its path of convergents outlined in Section 2.1, and which is illustrated in Figure 3.4.

The integers b_2, \dots, b_n of the expansion $[b_1, \dots, b_n]$ of a vertex x encode a set of directions to navigate the corresponding path $\langle \infty, v_1, \dots, v_n \rangle$; informally, to navigate the path you should upon reaching v_{i-1} take the ' $b_i/2$ th right turn' to get to v_i (which is a

convergents of an infinite even-integer continued fraction form an infinite simple path $\langle \infty, v_1, v_2, \dots \rangle$ in \mathcal{F}_∞ , and conversely each infinite path of this type is comprised of the convergents of an infinite even-integer continued fraction.

We end this section with a brief discussion of geodesic even-integer continued fractions. As in Chapter 2, we say that an even-integer continued fraction $x = [b_1, \dots, b_n]$ is geodesic if every other even-integer continued fraction expansion of x has at least n terms. Similarly, we say that an infinite even-integer continued fraction $[b_1, b_2, \dots]$ is geodesic if $[b_1, \dots, b_k]$ is a geodesic even-integer continued fraction for all $k = 1, 2, \dots$. It follows from Theorem 3.5 that every even-integer continued fraction is geodesic.

3.2 THE THEORY OF EVEN-INTEGER CONTINUED FRACTIONS

In this section we develop an elementary theory of even-integer continued fractions, proving fundamental results about the existence and uniqueness of even-integer continued fractions, and providing counterparts to some classical theorems from the theory of regular continued fractions. Many of the results in this section are unoriginal. The results in Section 3.2.2 in particular are similar to ones in [29] and [43]. Our method of proof using the geometric properties of the Farey graph \mathcal{F}_∞ , however, is original, and it is simple and elegant. Additionally, the proof of Theorem 3.13 is the first not to assume its counterpart for regular continued fractions, and the content of Section 3.2.6 is new.

3.2.1 INFINITE EVEN-INTEGER CONTINUED FRACTIONS

In this section we prove that every infinite even-integer continued fraction converges to either an irrational or a 1-rational number. There are several ways we could do this; we could, for example, invoke a more general theorem on the convergence of continued fractions, or we could use algebraic relationships between the convergents to estimate the distance between consecutive convergents. Our approach is to use the Farey graph \mathcal{F}_∞ to establish the following theorem.

Theorem 3.6. *Every infinite even-integer continued fraction converges to an irrational or a 1-rational.*

Proof. Let $[b_1, b_2, \dots]$ be an infinite even-integer continued fraction, and let γ be its corresponding path of convergents. First we will show that γ cannot accumulate at an ∞ -rational. Suppose, on the contrary, that γ does accumulate at a vertex x of \mathcal{F}_∞ . By applying an element of Γ_∞ to γ if necessary we can assume that $x \neq \infty$. Furthermore, by removing a finite number of terms from γ we can assume that it does not pass through x , since γ is simple. Let v be the initial vertex of γ .

Choose neighbours u and w of x such that $u < x < w$ and v lies outside the interval (u, w) , as shown in Figure 3.6. Edges of \mathcal{F}_∞ do not intersect in \mathbb{H} , so we see that because γ accumulates at x , it must pass through one of u , x and v . However, because \mathcal{F}_∞ is a tree, with only one exception, any path from v to a neighbour of x must pass through x . So providing we choose u and w such that neither of them are the exceptional neighbour, we see that γ passes through x . This contradicts the assumption, that γ accumulates at x , and so we see that γ cannot accumulate at an ∞ -rational.

It remains to show that γ accumulates at only one number. Suppose, in order to reach a contradiction, that γ accumulates at two numbers x and y , each of which is either irrational or a 1-rational,

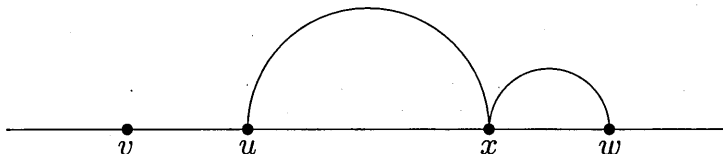


Figure 3.6: Two neighbours u and w of the vertex x , and another vertex v

and $x < y$. Since \mathcal{F}_∞ is a connected graph, there must be neighbouring vertices u and v of \mathcal{F}_∞ with u inside (x, y) and v outside. Then, since edges of \mathcal{F}_∞ do not intersect in \mathbb{H} , we see that because γ accumulates at both x and y , it must pass through at least one of u or v infinitely many times, which is impossible as γ is a simple path. Thus, contrary to our assumption, γ cannot accumulate at two numbers, and so it converges. The proof of Theorem 3.6 is now complete. \square

3.2.2 REPRESENTING REAL NUMBERS BY EVEN-INTEGERS CONTINUED FRACTIONS

It follows from Theorem 3.5 that every ∞ -rational number has a unique even-integer continued fraction expansion. We show now that every 1-rational number has precisely two even-integer continued fraction expansions, and that every irrational number has a unique even-integer continued fraction expansion. We begin this discussion by looking at even-integer continued fraction expansions of the number 1.

Lemma 3.7. *The number 1 has precisely two even-integer continued fraction expansions.*

Proof. We begin by giving two even-integer continued fraction expansions of the number 1. We can write

$$1 = [2, 2, 2, 2, \dots] = [0, -2, -2, -2, -2, \dots].$$

To check that the value x of the first continued fraction is 1 we observe that x must satisfy

$$x = 2 - \frac{1}{2 - \frac{1}{x}},$$

and the only solution of this equation is $x = 1$. The value of the second continued fraction can be obtained in a similar manner. The paths of convergents, α and β , that correspond to the even-integer continued fractions $[0, -2, -2, -2, -2, \dots]$ and $[2, 2, 2, 2, \dots]$ respectively, are shown in Figure 3.7.

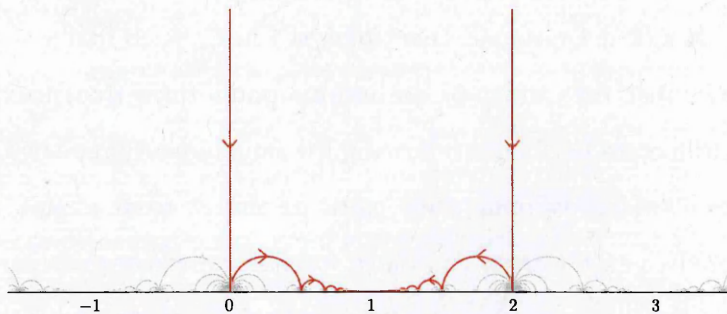


Figure 3.7: Two paths that converge to 1

It remains to prove that the two expansions that we have found are the only even-integer continued fraction expansions of 1. To see why this is so, observe that in the Farey graph \mathcal{F} , every single one of the vertices in these two paths is connected to 1 by an edge (in fact, they are the full collection of neighbours of 1 in \mathcal{F} – see Figure 3.3). Two such edges are shown in Figure 3.8, on either side of 1.

Suppose now that γ is an infinite path in \mathcal{F}_∞ from ∞ to 1. Aside from the initial vertex ∞ , this path must lie entirely to the left or entirely to the right of 1, because, as in the proof of Theorem 3.3, any path in \mathcal{F}_∞ that passes from one side to the other of 1 must pass through ∞ . Suppose that it lies to the left – the other case can be handled in a similar way. Then because edges in the Farey graph do

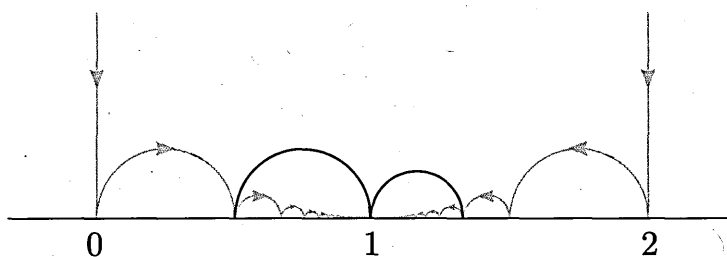


Figure 3.8: Two neighbours of 1

not intersect, γ must pass through all of the vertices of α . There is only one such path that does this, namely α itself, and so $\gamma = \alpha$. \square

Corollary 3.8. *Every 1-rational has precisely two even-integer continued fraction expansions.*

Proof. Let x be a 1-rational. Then there is f in Γ_∞ such that $x = f(1)$. It follows that $f(\alpha)$ and $f(\beta)$ are infinite paths from $f(\infty)$ to x . By connecting ∞ to $f(\infty)$, and removing the finitely many repeated vertices if necessary, we obtain two paths α' and β' from ∞ to x , and therefore two even-integer continued fraction expansions of x .

To see that these are the only two even-integer continued fraction expansions of x , assume that there is a third path γ from ∞ to x . We obtain a path γ' from $f(\infty)$ to x by connecting γ to $f(\infty)$ and removing, if necessary, the finitely many repeated vertices. Now γ' cannot be identical to either of $f(\alpha)$ or $f(\beta)$, else γ must have infinitely many vertices in common with either α' or β' . But then $f^{-1}(\gamma')$ is a third path from ∞ to 1, contradicting Lemma 3.7. Therefore there can be only two even-integer continued fraction expansions of x . \square

We have shown that every 1-rational has precisely two even-integer continued fraction expansions. We will see in Section 3.2.4 that the sequence of coefficients of each of these is eventually constant with value 2 or -2 . Conversely, we will see that any real number with an even-integer continued fraction expansion whose sequence of coefficients is eventually constant with value 2 or -2 is a 1-rational.

Let us now consider the even-integer continued fraction expansions of an irrational number. We begin by showing that every irrational number x has at least one even-integer continued fraction expansion. One way to do this is to use an algorithm of a similar type to Euclid's algorithm. However, we prefer to justify the existence of an expansion using the Farey graph and tree.

We define a *Farey interval* to be a real interval whose endpoints are neighbouring vertices in the Farey graph \mathcal{F} . If $[a/b, c/d]$ is a Farey interval (where, as usual, the fractions are given in reduced form), then it is easily seen that $[a/b, (a+c)/(b+d)]$ and $[(a+c)/(b+d), c/d]$ are both Farey intervals – let us call them the *Farey subintervals* of $[a/b, c/d]$. Now, any irrational x belongs to a Farey interval $[n, n+1]$, where n is the integer part of x , and by repeatedly choosing Farey subintervals, we can construct a nested sequence of Farey intervals that contains x in its intersection. The width of one of these intervals $[a/b, c/d]$ is

$$\left| \frac{a}{b} - \frac{c}{d} \right| = \left| \frac{ad - bc}{bd} \right| = \frac{1}{bd},$$

so we see that the sequence of widths of this nested sequence of Farey intervals converges to 0.

Let us now restrict our attention to those infinitely many Farey intervals $I_1 \supset I_2 \supset \dots$ from the sequence for which one of the endpoints of I_n is a 1-rational v_n (and the other endpoint u_n must then be an ∞ -rational). Let γ_n be the unique path from ∞ to u_n in \mathcal{F}_∞ . Any path in \mathcal{F}_∞ from ∞ to a vertex inside I_{n-1} must pass through u_{n-1} (because u_{n-1} and v_{n-1} are neighbours in \mathcal{F} , as illustrated in Figure 3.9, and edges of \mathcal{F} do not intersect). Therefore γ_{n-1} is a subpath of γ_n . It follows that there is a unique infinite path γ that contains every path γ_n as a subpath. The path γ passes through all the vertices u_n , which accumulate at x , so γ must converge to x . It follows

that any irrational number x has at least one even-integer continued fraction expansion.

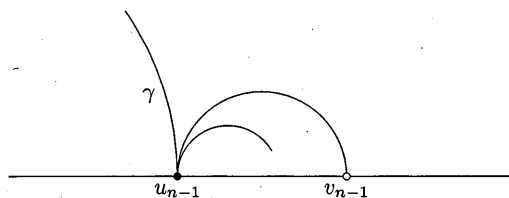


Figure 3.9: The path γ passes through u_{n-1}

We have seen that every real number x has at least one even-integer continued fraction expansion. In fact, unless x is 1-rational, it has a unique even-integer continued fraction expansion. This fact is proved in the following theorem, which collects together many of the results presented in this section.

Theorem 3.9.

1. *The value of any finite even-integer continued fraction is an ∞ -rational, and each ∞ -rational has a unique finite even-integer continued fraction expansion.*
2. *The value of an infinite even-integer continued fraction is either irrational or a 1-rational, and*
 - a) *each irrational has a unique infinite even-integer continued fraction expansion,*
 - b) *each 1-rational has exactly two infinite even-integer continued fraction expansions.*

Proof. We have seen already that every real number has an even-integer continued fraction expansion. Statement 1 follows immediately from Theorem 3.4 and Theorem 3.5. The first part of statement 2 follows from Theorem 3.6, and statement b) follows from Corollary 3.8; it remains only to discuss the uniqueness assertion of statement a).

Suppose then that α and β are two infinite paths from ∞ to a real number x . The two paths may coincide for a certain number of vertices: let w be the final vertex for which they do so. Choose an element f of Γ_∞ such that $f(w) = \infty$. Let α' and β' be the paths obtained from $f(\alpha)$ and $f(\beta)$, respectively, after removing all vertices that occur before ∞ . Then α' and β' are infinite paths from ∞ to $f(x)$, such that the second vertex u of α' is distinct from the second vertex v of β' . The vertices u and v are even integers, so there is an odd integer q (a 1-rational) that lies between them on the real line. Neither α' nor β' may cross q , as discussed in the proof of Theorem 3.3, and since they converge to the same value, that value must be q . Therefore $f(x)$ is a 1-rational, so x is also a 1-rational. This proves statement *a*), completing the proof of the theorem. \square

We end this section by noting that an even-integer continued fraction expansion of a real number x can be obtained by applying the nearest even-integer continued fraction algorithm, which is the analogue for even-integer continued fractions of the nearest-integer algorithm used to compute geodesic Rosen continued fractions in Chapter 2. In fact, with care, the arguments presented in Section 2.2 can be adapted to give a geometric interpretation of the nearest even-integer algorithm, and prove that it gives an even-integer continued fraction expansion of any real number. It follows from Theorem 3.9 that all algorithms for computing the even-integer continued fraction expansion of an irrational or 1-rational number are equivalent. It follows that the method presented in this section for obtaining an even-integer continued fraction expansion of an irrational number x is equivalent to the nearest-integer algorithm.

3.2.3 THE RELATIONSHIP BETWEEN THE REGULAR AND EVEN-INTEGER CONTINUED FRACTION EXPANSIONS OF A REAL NUMBER

In [29] the authors describe how to obtain the even-integer continued fraction expansion of a real number from its regular continued fraction expansion. Two processes, *singularization* and *insertion*, are used. Here we explain informally how these processes can be illuminated using paths in \mathcal{F}_∞ , without engaging with the details of the arguments presented in [29].

We suppose first that x is ∞ -rational; later in the section we will consider the case in which x is 1-rational or irrational. We begin by considering the regular continued fraction expansion of x . Since the point x is a vertex of the Farey graph \mathcal{F} , we can define the 3-chain from ∞ to x , P_1, \dots, P_n (see Section 2.3.1). We use the notation described at the start of Section 2.3.2, which we illustrate again in Figure 3.10.

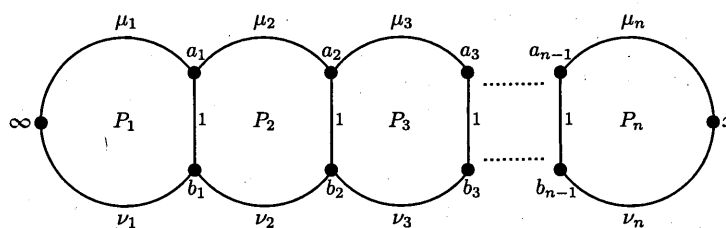


Figure 3.10: A 3-chain from ∞ to x

By Theorem 2.2, the regular continued fraction expansion $[b_1, \dots, b_n]$ of x can be viewed as a path in \mathcal{F} between ∞ and x . In fact, this path lies entirely within the 3-chain from ∞ to x . To see this, we use that fact that the regular continued fraction expansion of a real number is obtained using Euclid's algorithm. Geometrically, Euclid's algorithm is equivalent to applying to the point ∞ some sequence of the maps α_x and β_x that were defined in Section 2.2.1; this can

be proved in much the same way as it is proved in Section 2.2.2 that the nearest-integer algorithm is equivalent geometrically to iterating the map α_x . By the definition of the maps α_x and β_x , it follows that the path of convergents corresponding to the regular continued fraction expansion of x lies entirely within the 3-chain from ∞ to x . By Lemma 2.3, we know that because of the alternating signs of the coefficients b_2, \dots, b_n of the regular continued fraction expansion of x , the corresponding path of convergents ‘zig-zags’ through the 3-chain from ∞ to x (see also [48]). Figure 3.11 illustrates an example of a 3-chain from ∞ to an ∞ -rational x , along with the path of convergents corresponding to the regular continued fraction expansion of x .

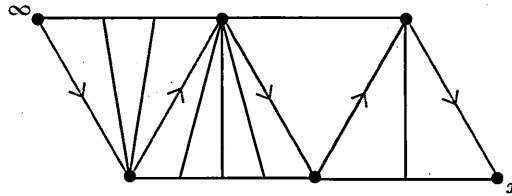


Figure 3.11: The path of convergents corresponding to the regular continued fraction expansion $[0, -3, 4, -1, 2]$ of an ∞ -rational x within the 3-chain from ∞ to x

In the following, we will describe an algorithm that allows us to move from the path of convergents corresponding to the regular continued fraction expansion of x to the path of convergents corresponding to the even-integer continued fraction expansion of x . We need only consider the finite portion of \mathcal{F} lying in the 3-chain from ∞ to x , since it follows from Lemma 3.2 that there is a path between ∞ and x in this 3-chain that passes only through vertices of \mathcal{F}_∞ (see also the proof of Theorem 3.3), and this path must be the unique path in \mathcal{F}_∞ between ∞ and x .

We now describe the two processes of singularization and insertion. *Singularization* involves replacing a section of path traversing two edges of a face with a path traversing the third edge of the face – see Figure 3.12. This corresponds to removing a coefficient $b_i = \pm 1$

turn if b_i is negative); this idea was formalised in Section 3.1.3. Within the 3-chain from ∞ to x , this means that there are b_i triangles meeting at v_{i-1} that lie between the edges $\{v_{i-2}, v_{i-1}\}$ and $\{v_{i-1}, v_i\}$. This is illustrated in Figure 3.11.

If b_i is even for all i then we have nothing to prove. Otherwise, let k be the smallest positive integer such that b_k is odd. If $b_{k+1} = \pm 1$ we apply singularisation to remove the vertex v_k . This changes the parity of b_k (see Figure 3.12). If $b_{k+1} \neq \pm 1$, we first apply the insertion process $b_{k+1} - 1$ times, as illustrated in Figure 3.14. This gives us the new sequence of coefficients

$$b_k \pm 1, \underbrace{\pm 2, \dots, \pm 2}_{b_{k+1} - 1 \pm 2s}, \pm 1, \pm 1.$$

We apply singularisation to the second ± 1 , to obtain the sequence

$$b_k \pm 1, \underbrace{\pm 2, \dots, \pm 2}_{b_{k+1} \pm 2s}.$$

This process is illustrated in Figure 3.14.

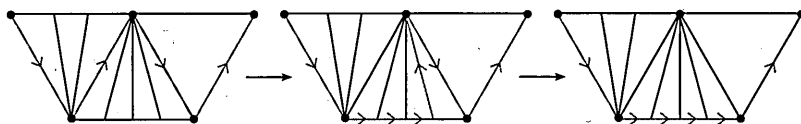


Figure 3.14: Applying insertion $b_{k+1} - 1$ times followed by singularisation

We can repeat this process, and at each stage the number of coefficients of the continued fraction expansion corresponding to the path strictly increases. Correspondingly, the number of vertices of the path that lie in \mathcal{F}_∞ strictly increases. Eventually, therefore, we are left only with the final vertex of the path, say v_l , to consider. But we have ensured, using the singularization and insertion processes, that v_{l-1} is ∞ -rational, and by assumption v_l is ∞ -rational, so we have obtained a path in \mathcal{F}_∞ between ∞ and x . \square

We now consider the regular and even-integer continued fraction expansions of an irrational number x . Using the method described in Section 2.3.1, we can define an infinite 3-chain from ∞ to x in which both paths of convergents corresponding to the regular and even-integer continued fraction expansions of x lie. We can apply the algorithm described above in exactly the same way, except that the algorithm will not terminate. It is clear, however, that the algorithm produces the unique even-integer continued fraction expansion of x .

When x is a 1-rational the 3-chain from ∞ to x is finite. When applying the algorithm, however, we reach a point where we have a path $\langle v_1, \dots, v_n \rangle$ where $\infty = v_1$, $v_n = x$ and v_1, \dots, v_{n-1} lie in \mathcal{F} , but x is not a vertex of \mathcal{F}_∞ . We apply the insertion procedure indefinitely to get an infinite sequence $\pm 2, \pm 2, \dots$; this is the even-integer continued fraction expansion of x .

3.2.4 SERRET'S THEOREM ON CONTINUED FRACTIONS

This section introduces a counterpart for a well-known theorem of Serret on regular continued fractions. Before stating our theorem, we introduce an equivalence relation on the extended real line $\hat{\mathbb{R}}$. Recall that the extended theta group $\tilde{\Gamma}_\infty$ is the group generated by Γ_∞ and the Möbius transformation $\kappa(z) = -\bar{z}$. The group $\tilde{\Gamma}_\infty$ acts on the upper half-plane \mathbb{H} as a group of automorphisms of \mathcal{F}_∞ ; $\tilde{\Gamma}_\infty$ also acts on the ideal boundary $\hat{\mathbb{R}}$ of \mathbb{H} . We say that two real numbers x and y are *equivalent* under the action of $\tilde{\Gamma}_\infty$ if they lie in the same orbit under this action, that is, if there is $f \in \tilde{\Gamma}_\infty$ such that $y = f(x)$. It is immediate that the set of ∞ -rationals is an equivalence class under this relation, as is the set of 1-rationals.

We are now ready to state our version of Serret's theorem for even-integer continued fractions. It is similar to [29, Theorem 1], although

in that paper even-integer continued fractions are defined differently, making the statement slightly different.

Theorem 3.11. *Two real numbers x and y that are not ∞ -rationals are equivalent under $\tilde{\Gamma}_\infty$ if and only if there are positive integers k and l such that the even-integer continued fraction expansions of x and y ,*

$$x = [a_1, a_2, \dots] \quad \text{and} \quad y = [b_1, b_2, \dots],$$

either satisfy $a_{k+i} = b_{l+i}$ for $i = 1, 2, \dots$ or $a_{k+i} = -b_{l+i}$ for $i = 1, 2, \dots$

Serret's theorem for regular continued fraction expansions is similar, but uses an extension of the modular group rather than the theta group, and the possibility $a_{k+i} = -b_{l+i}$ for $i = 1, 2, \dots$ is absent.

Crucial to the proof of this theorem is the following lemma.

Lemma 3.12. *If a real number x has an even-integer continued fraction expansion $[b_1, b_2, \dots]$, then an even-integer continued fraction expansion of $-x$ is $[-b_1, -b_2, \dots]$.*

There is no obvious analogue of this lemma for regular continued fractions because the coefficients of regular continued fractions are (almost) all positive.

Lemma 3.12 can be proven with the Farey tree by observing that the paths from ∞ to x and from ∞ to $-x$ are reflections of each other in the imaginary axis. However, in this case, we will prove the lemma using Möbius transformations, in a similar manner to the proof of Lemma 2.26.

Proof of Lemma 3.12. For each even integer b , let $T_b(z) = b - 1/z$, that is, $T_b = \tau^{b/2}\sigma$. Observe that $\kappa T_b \kappa = T_{-b}$. Since $[b_1, b_2, \dots]$ is an even-integer continued fraction expansion of x ,

$$T_{b_1} T_{b_2} \cdots T_{b_k}(\infty) \rightarrow x$$

as $k \rightarrow \infty$. Now

$$\begin{aligned} T_{-b_1} T_{-b_2} \cdots T_{-b_k}(\infty) &= \kappa T_{b_1} T_{b_2} \cdots T_{b_k} \kappa(\infty) \\ &= \kappa T_{b_1} T_{b_2} \cdots T_{b_k}(\infty). \end{aligned}$$

So $T_{-b_1} T_{-b_2} \cdots T_{-b_k}(\infty) \rightarrow \kappa(x) = -x$ as $k \rightarrow \infty$. Therefore $[-b_1, -b_2, \dots]$ is an even-integer continued fraction expansion of $-x$. \square

Let us now prove Theorem 3.11.

Proof of Theorem 3.11. Suppose first that $y = g(x)$, where $g \in \tilde{\Gamma}_\infty$. We wish to prove that there are positive integers k and l such that $a_{k+i} = b_{l+i}$ for $i = 1, 2, \dots$ or $a_{k+i} = -b_{l+i}$ for $i = 1, 2, \dots$. Since $\tilde{\Gamma}_\infty$ is generated by the transformations $\kappa(z) = -\bar{z}$, $\sigma(z) = 1/z$ and $\tau(z) = z + 2$, it suffices to prove the assertion when g is each of κ , σ , τ and τ^{-1} . It is straightforward to do so when g is one of the final three transformations, and the remaining case when g is κ is an immediate consequence of Lemma 3.12.

For the converse, suppose that $x = [a_1, a_2, \dots]$, $y = [b_1, b_2, \dots]$ and either (i) $a_{k+i} = b_{l+i}$ for $i = 1, 2, \dots$, or (ii) $a_{k+i} = -b_{l+i}$ for $i = 1, 2, \dots$. By replacing x by $-x$ if necessary, and invoking Lemma 3.12, we can assume that (i) holds. Observe that, defining $T_b(z) = b - 1/z$ for each even integer b ,

$$x = T_{a_1} \cdots T_{a_k}([a_{k+1}, a_{k+2}, \dots])$$

and

$$y = T_{b_1} \cdots T_{b_l}([b_{l+1}, b_{l+2}, \dots]).$$

Hence $y = T_{b_1} \cdots T_{b_l} T_{a_k}^{-1} \cdots T_{a_1}^{-1}(x)$, so x and y are equivalent under $\tilde{\Theta}$. This completes the proof of Theorem 3.11. \square

We saw in the proof of Lemma 3.7 that the number one has precisely two even-integer continued fraction expansions, and that these are

$$1 = [2, 2, 2, 2, \dots] = [0, -2, -2, -2, -2, \dots].$$

It follows from Theorem 3.11 that a real number x is a 1-rational if and only if it has an even-integer continued fraction expansion $[b_1, b_2, \dots]$ such that the coefficients b_i are eventually all equal to either 2 or -2 . Furthermore, it is straightforward to check that the two even-integer continued fractions

$$[b_1, \dots, b_n, 2, 2, 2, \dots] \quad \text{and} \quad [b_1, \dots, b_{n-1}, b_n + 2, -2, -2, -2, \dots],$$

if $b_n > 0$, and

$$[b_1, \dots, b_n, 2, 2, 2, \dots] \quad \text{and} \quad [b_1, \dots, b_{n-1}, b_n - 2, -2, -2, -2, \dots],$$

if $b_n < 0$, have the same value, and so the two even-integer continued fraction expansions of x referred to in Corollary 3.8 are of these forms.

3.2.5 PERIODIC EVEN-INTEGER CONTINUED FRACTIONS

Of special interest amongst regular continued fractions are periodic regular continued fractions, because of their close connection to quadratic irrationals. This connection was made explicit in a classical theorem of Lagrange [31].

Theorem (Lagrange). *The real numbers whose regular continued fraction expansions are periodic are precisely the quadratic irrationals.*

In this section we prove an analogous theorem for even-integer continued fractions. We say that an even-integer continued fraction is

periodic if its sequence of coefficients is periodic, that is, if it can be written in the form

$$[b_1, b_2, \dots, b_n, \overline{a_1, a_2, \dots, a_m}],$$

where the string a_1, \dots, a_m repeats infinitely. We say that the expansion is *purely periodic* if the length of the sequence b_1, \dots, b_n has length zero.

Theorem 3.13. *The irrational numbers whose even-integer continued fraction expansions are periodic are precisely the quadratic irrationals.*

Notice that the word ‘irrational’ cannot be replaced with ‘real’ in the statement of Theorem 3.13, since we know from Corollary 3.8 and Theorem 3.11 that every 1-rational has a periodic even-integer continued fraction expansion.

There are several ways in which Theorem 3.13 can be proved. Firstly, the theorem can be deduced from Lagrange’s theorem for periodic regular continued fractions; this is the approach taken in [29]. Geometric methods used by Hatcher [20] and Conway [10] to prove Lagrange’s theorem for periodic regular continued fractions can, with care, be adapted to prove Theorem 3.13. Here, however, we present a proof using the techniques introduced in this thesis. Key to our proof is the following theorem, which relates quadratic irrationals to hyperbolic elements of Γ_∞ .

Theorem 3.14. *Let x and y be distinct real numbers. Then x and y are conjugate quadratic irrationals if and only if they are the fixed points of some hyperbolic element f of Γ_∞ .*

Proof. Let x and y be the fixed points of the hyperbolic element

$$f(z) = \frac{az + b}{cz + d}.$$

of Γ_∞ . The fixed points of f are

$$\frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2}. \quad (3.1)$$

Since f is hyperbolic, $(a+d)^2 > 4$, and so both roots of (3.1) are real. It remains to show that these roots are irrational. Suppose not, then $k = \sqrt{(a+d)^2 - 4}$ is rational, and since k is the square root of an integer, k must be an integer itself. Now $k^2 + 4 = (a+d)^2$, and so $(a+d-k)(a+d+k) = 4$. But the equation $(a+d-k)(a+d+k) = 4$ has no integer solutions for k , which means that k , and hence the roots of (3.1), are irrational. It follows that the fixed points of f are conjugate quadratic irrationals.

We prove the converse statement using the geometry of \mathcal{F}_∞ . Suppose that x and y are quadratic irrationals that are the conjugate solutions to the quadratic equation $Az^2 + Bz + C = 0$ where $A > 0$, $\gcd(A < B < C) = 1$ and $B^2 - 4AC$ is real and not an integer. Let ℓ be the hyperbolic geodesic between x and y . Since x and y are neither vertices of \mathcal{F}_∞ nor 1-rationals, there are infinitely many edges of \mathcal{F}_∞ that intersect ℓ . If not, then after applying an element of Γ_∞ , we can assume that $x > 0$ and that there is no edge of \mathcal{F}_∞ with endpoints a and b satisfying $0 < a < x < b$; it follows that x is either ∞ -rational or an odd integer, a contradiction.

Let δ denote the edge of \mathcal{F}_∞ between 0 and ∞ . The edges of \mathcal{F}_∞ intersecting ℓ are the images of δ under some collection of distinct elements $\{f_i\}$ of Γ_∞ . Let $x_i = f_i^{-1}(x)$ and $y_i = f_i^{-1}(y)$. It can easily be shown that each pair x_i and y_i is a conjugate pair of solutions to a quadratic equation $A_i z^2 + B_i z + C_i = 0$, with $B_i^2 - 4A_i C_i = B^2 - 4AC = \Delta$ for some fixed integer Δ . Now, for all i , one out of x_i and y_i is positive, while the other is negative, and so $C_i/A_i = x_i y_i < 0$. It follows that $A_i C_i < 0$ and there can be only finitely many solutions for A_i , B_i and C_i to the equation $B_i^2 - 4A_i C_i = \Delta$. In particular, we

can find j and k with $f_j \neq f_k$ but $f_j^{-1}(\ell) = f_k^{-1}(\ell)$. Then $f = f_j f_k^{-1}$ fixes ℓ . Since f is not the identity, it follows that f is a hyperbolic element of Γ_∞ with fixed points x and y . \square

We are now ready to prove the first half of Theorem 3.13.

Lemma 3.15. *Let x be an irrational number with a periodic even-integer continued fraction expansion. Then x is a quadratic irrational.*

Proof. Let x be an irrational number with a periodic even-integer continued fraction expansion $[b_1, b_2, \dots, b_n, \overline{a_1, a_2, \dots, a_m}]$. By Theorem 3.14 it is enough to show that x is a fixed point of a hyperbolic element of Γ_∞ . Note that Γ_∞ contains no elliptic elements, and x is irrational so cannot be the fixed point of a parabolic element of Γ_∞ .

For each even integer we define the map $T_b(z) = b - 1/z$. Let $g = T_{b_1} \cdots T_{b_n}$ and $f = T_{a_1} \cdots T_{a_m}$, so that $x = \lim_{k \rightarrow \infty} g f^k(\infty)$. Write $h = g f g^{-1}$, then $x = \lim_{k \rightarrow \infty} h^k(g(\infty))$. If $g(\infty)$ is a fixed point of h , then $x = g(\infty)$ is rational, a contradiction, so h does not fix $g(\infty)$. Therefore, $h^i(g(\infty))$ converges to the attracting fixed point of h , η^+ , and so $x = \eta^+$ is the attracting fixed point of a hyperbolic element of Γ_∞ . \square

To prove the converse statement of Theorem 3.13, we use the following lemma.

Lemma 3.16. *Let f be a hyperbolic element of Γ_∞ . Then $f = h g h^{-1}$ for some $h = \tau^{a_1} \sigma \cdots \tau^{a_m} \sigma$ and $g = \tau^{b_1} \sigma \cdots \sigma \tau^{b_n}$, where $m \geq 0$ and $n > 0$ are integers, and the sequences of even-integers a_1, \dots, a_m and b_2, \dots, b_{n-1} satisfy $a_2, \dots, a_m \neq 0$, $b_1, \dots, b_n \neq 0$ and $b_1 \neq -b_n$.*

Proof. Since f is an element of Γ_∞ , we can write

$$f = \tau^{c_1} \sigma \cdots \sigma \tau^{c_l},$$

for even integers c_1, \dots, c_l , with $b_i \neq 0$ for $i = 2, \dots, l-1$. If $c_1 \neq c_l$, then we can take h to be the identity transformation, and we are done. Otherwise, there is some k such that $c_i = -c_l - i + 1$ for all $i = 1, \dots, k$, but $c_{k+1} \neq c_{l-k}$; if there was no such k , the f can be written

$$f = \tau^{c_1} \sigma \dots \tau^{c_j} \sigma \tau^{-c_j} \dots \sigma \tau^{-c_1},$$

for some j . But then f is conjugate to the parabolic Möbius transformation σ , and is therefore parabolic itself, which is a contradiction. Therefore, we can write f in the form

$$f = \tau^{c_1} \sigma \dots \sigma \tau^{c_k} \sigma \tau^{c_{k+1}} \sigma \dots \sigma \tau^{c_{l-k}} \sigma \tau^{-c_k} \sigma \dots \sigma \tau^{-c_1}.$$

Letting $h = \tau^{c_1} \sigma \dots \tau^{c_k} \sigma$ and $g = \tau^{c_{k+1}} \sigma \dots \sigma \tau^{c_{l-k}}$, we have $f = hgh^{-1}$ with h and g in the required form. \square

Proof of Theorem 3.13. By Lemma 3.15, if x is an irrational number with a periodic even-integer continued fraction expansion then x is a quadratic irrational. To prove the converse implication, suppose that x is a quadratic irrational, with conjugate quadratic irrational y . By Theorem 3.14, x and y are the fixed points of a hyperbolic element f of Γ_∞ . By replacing f with f^{-1} if necessary, we may assume that x is the attracting fixed point of f . By Lemma 3.16, we can write $f = hgh^{-1}$ where $h = \tau^{a_1} \sigma \dots \tau^{a_m} \sigma$ and $g = \tau^{b_1} \sigma \dots \sigma \tau^{b_n}$ where $m \geq 0$, $n > 0$, and the sequences of even integers a_1, \dots, a_m and b_2, \dots, b_{n-1} satisfy $a_2, \dots, a_m \neq 0$, $b_1, \dots, b_n \neq 0$ and $b_1 \neq -b_n$.

Suppose that $b_1 \neq 0$; the other two cases in which $b_1 = 0$ and $b_n \neq 0$, or $b_1 = b_n = 0$ are similar. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} f^n(\infty) &= \lim_{n \rightarrow \infty} h g^n h^{-1}(\infty) \\ &= [a_1, \dots, a_m, b_1, \overline{b_2, \dots, b_{n-1}, b_n + b_1}]. \end{aligned}$$

Since ∞ is not the repelling fixed point of f , $\lim_{n \rightarrow \infty} f^n(\infty) = x$, and we see that x has a periodic even-integer continued fraction expansion. \square

3.2.6 AN ALTERNATIVE CHARACTERISATION OF THE CONVERGENTS OF AN EVEN-INTEGER CONTINUED FRACTION

Here we describe an alternative way to characterise the convergents of the even-integer continued fraction expansion of any irrational number x . The characterisation is new, and will play an important role in the work of Section 3.3. It can easily be adapted to allow x to be rational; we omit the details here.

Theorem 3.17. *A finite ∞ -rational v is a convergent of the even-integer continued fraction expansion of an irrational number x if and only if there is a 1-rational w adjacent to v in the Farey graph \mathcal{F} such that x lies between v and w on the real line.*

The second statement of the theorem is illustrated in Figure 3.15. To see that in this situation v must be a convergent of the even-integer continued fraction expansion of x , notice that any vertex u of \mathcal{F} that is sufficiently close to x on the real line is separated from ∞ by the edge incident to v and w . It follows, since edges of \mathcal{F} do not intersect one another, that any path from ∞ to u in \mathcal{F} must pass through one of v or w – and if the path lies in \mathcal{F}_∞ , then it must pass through v . In

particular, this demonstrates that v is a convergent of the even-integer continued fraction expansion of x .

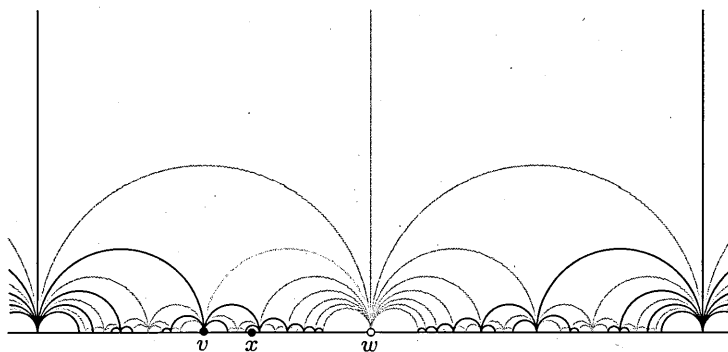


Figure 3.15: The irrational number x lies between the ∞ -rational v and the 1-rational w

The converse implication of Theorem 3.17 is a direct consequence of the following lemma, which is a slightly stronger statement.

Lemma 3.18. *Let v_i and v_{i+1} be two consecutive convergents of the even-integer continued fraction expansion of an irrational x , in that order. Then there is a 1-rational w adjacent to each of v_i and v_{i+1} in the Farey graph such that both v_{i+1} and x lie between v_i and w on the real line.*

Proof. Since v_i and v_{i+1} are adjacent in \mathcal{F}_∞ , they are also adjacent in \mathcal{F} . There are two other vertices of \mathcal{F} that are adjacent to both v_i and v_{i+1} , precisely one of which, say w , does *not* lie between v_i and v_{i+1} on the real line. Let γ be the path of convergents of the even-integer continued fraction expansion of x . If γ enters the interval between v_i and w , then by Lemma 2.5 it must pass through v_i to get there, and it cannot then leave the interval. Similarly, if γ enters the interval between v_{i+1} and w , then it must pass through v_{i+1} to get there, and it cannot then leave the interval. It follows that v_i cannot lie in the interval between v_{i+1} and w , else the path γ would pass through v_{i+1} before passing through v_i . So v_{i+1} lies in the interval between v_i and w (as illustrated in Figure 3.16), and x lies in that interval too. \square

This completes the proof of Theorem 3.17.

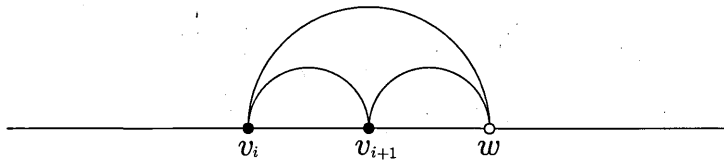


Figure 3.16: A triangle in the Farey graph

3.3 EVEN-INTEGERS CONTINUED FRACTIONS AND DIOPHANTINE APPROXIMATION

One of the principal uses of continued fractions is in the field of Diophantine approximation, which is concerned with the approximation of real numbers by rationals. We call a rational number a/b a *strong approximant* of a real number x if for each rational number c/d such that $d \leq b$, we have

$$|bx - a| \leq |dx - c|,$$

with equality if and only if $c/d = a/b$. It is a well-known theorem of Lagrange (see [27, Theorems 16 and 17]) that the strong approximants of a real number x are precisely the regular continued fraction convergents of x . It was observed by Schweiger [50] that the convergents of even-integers continued fractions are not necessarily strong approximants and, in fact, demonstrate poor approximation properties in general. When restricting to the approximation of real numbers by ∞ -rational numbers, however, the convergents of even-integers continued fractions are strong approximants; precisely what this means and why it is true is discussed in Section 3.3.2.

A second important part of Diophantine approximation is finding bounds on the accuracy of the approximation of a real number by a

rational. It was proved by Hurwitz [24] that for any irrational number x there are infinitely many reduced rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2},$$

and that the constant $1/\sqrt{5}$ is the best possible in the sense that if we replace $1/\sqrt{5}$ with any number less than $1/\sqrt{5}$ then there is some x for which there are only finitely many rational numbers p/q such that this inequality holds. The number $1/\sqrt{5}$ is called the *Hurwitz constant*. In Section 3.3.3 we obtain an analogous result for the approximation by ∞ -rationals.

To obtain results in the approximation of real numbers by ∞ -rational numbers we are required to use a different technique to those used so far in this thesis; we introduce *Ford circles*, as discussed in Section 1.2.3. We begin by discussing the basic geometric properties of Ford circles.

3.3.1 FORD CIRCLES

Recall that given a reduced rational $x = a/b$, the *Ford circle* C_x is the horocycle based at x with Euclidean radius $\text{rad}[C_x] = 1/(2b^2)$. The Ford circle C_∞ is the line $\text{Im}(z) = 1$ in \mathbb{C} together with the point ∞ . We will begin by discussing the geometry of Ford circles in more detail than we did in Section 1.2.3.

Simple Euclidean geometry (see Figure 3.17) tells us that two horocycles with radii r and s and distinct base points x and y intersect if and only if

$$|x - y|^2 \leq 4rs,$$

with equality if and only if the two horocycles are tangent. It follows that two Ford circles C_x and C_y , where $x = a/b$ and $y = c/d$, are tangent if and only if $|ad - bc| = 1$, and if they are not tangent then they are completely disjoint. Therefore, the full collection of Ford cir-

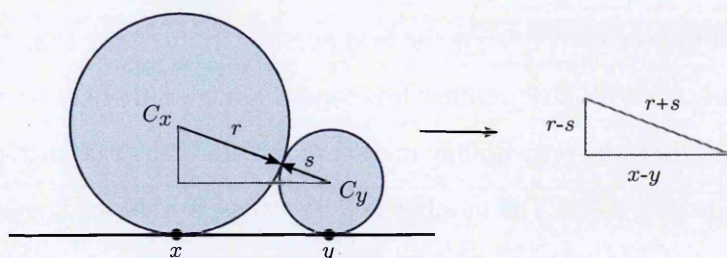


Figure 3.17: Two tangent horocycles

cles forms a model for the abstract graph underlying the Farey graph \mathcal{F} ; Ford circles represent the vertices of the graph, and two Ford circles are tangent if and only if their base points are joined by an edge of \mathcal{F} . A part of the collection of Ford circles superimposed with the Farey graph \mathcal{F} is shown in Figure 1.4. Similarly, the collection of Ford circles based at ∞ -rational numbers is a model of the abstract graph underlying the Farey graph \mathcal{F}_∞ ; this model, with the Farey graph \mathcal{F}_∞ superimposed, is illustrated in Figure 3.18.

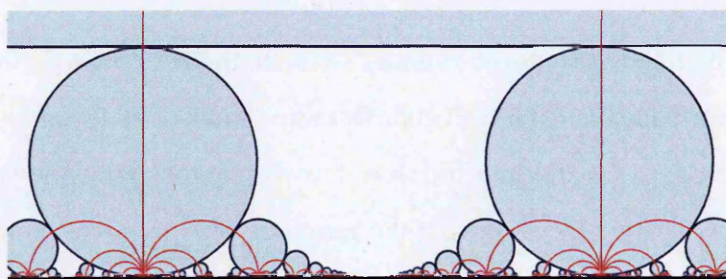


Figure 3.18: Ford circles based at the ∞ -rationals with the Farey graph \mathcal{F}_∞ superimposed

It is often convenient to view the collection of Ford circles based at ∞ -rational numbers as a subset of the full collection of Ford circles, as illustrated in Figure 3.19.

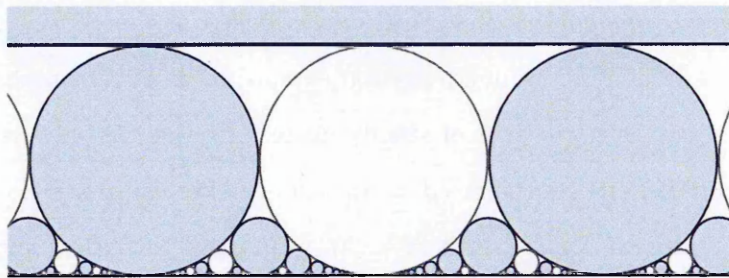


Figure 3.19: The Ford circles based at ∞ -rationals as a subset of the full collection of Ford circles

It can be easily shown that the full collection of Ford circles is invariant under the action of the modular group Γ . It follows that the collection of Ford circles based at ∞ -rationals is invariant under the theta group Γ_∞ .

Theorem 3.19. *Let x be a real number and v_1, v_2, \dots be the sequence of convergents of an even-integer continued fraction expansion of x . Then for all $i = 1, 2, \dots$,*

1. C_{v_i} is based at a ∞ -rational;
2. C_{v_i} is tangent to $C_{v_{i+1}}$;
3. $\text{rad}[C_{v_i}] > \text{rad}[C_{v_{i+1}}]$, and $\text{rad}[C_{v_i}] \rightarrow 0$ as $i \rightarrow \infty$.

Proof. The proof of statements 1 and 2 follow from [Theorem 3.4](#). By [Lemma 3.18](#) there is a 1-rational w adjacent to both v_i and v_{i+1} such that v_{i+1} lies between v_i and w . Writing v_i and w as reduced rationals $v_i = a/b$ and $w = c/d$, we see that since the three rationals a/b , c/d and $(a+b)/(c+d)$ are pairwise neighbours in \mathcal{F} , so are the three vertices v_i , w and v_{i+1} , we must have $v_{i+1} = (a+b)/(c+d)$, which is a reduced rational. Therefore the denominator of v_{i+1} is strictly less than the denominator of v_i , and so $\text{rad}[C_{v_i}] > \text{rad}[C_{v_{i+1}}]$. Furthermore, since $c+d \geq d+1$, $\text{rad}[C_{v_i}] \rightarrow 0$ as $i \rightarrow \infty$, proving statement

3. □

[Theorem 3.19](#) tells us that Ford circles based at the convergents of an even-integer continued fraction expansion of a real number x form a chain of horocycles of strictly decreasing size that lies in the collection of Ford circles based at ∞ -rationals. We call this the *even-integer continued fraction chain* of x . An example of a part of such a chain is illustrated in [Figure 3.20](#).

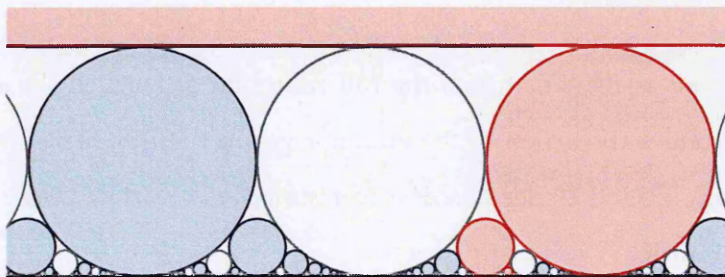


Figure 3.20: The even-integer continued fraction chain of $x = [2, 2, -2, -2]$

The continued fraction chain of a real number x encodes the information given by the path of convergents of x . In addition, we shall see that the distance between two convergents, and the distance between convergents and other real numbers, can be computed from the radii of the Ford circles. This allows us to use properties of Ford circles to prove theorems about the approximation of real numbers by ∞ -rationals in [Section 3.3.2](#) and [Section 3.3.3](#).

3.3.2 STRONG APPROXIMATION

In this section we prove an analogue for even-integer continued fractions of the classical result of Lagrange on strong approximation. We call an ∞ -rational a/b a *strong ∞ -approximant* of a real number x if for each ∞ -rational c/d such that $d \leq b$, we have

$$|bx - a| \leq |dx - c|,$$

with equality if and only if $c/d = a/b$. We prove the following theorem.

Theorem 3.20. *An ∞ -rational is a strong ∞ -approximant of a irrational number x if and only if it is a convergent of the even-integer continued fraction expansion of x .*

This result appears to be new, although an analogous result for a similar class of continued fractions is given in [43]. The techniques used in [43] are algebraic, but can be illuminated using the approach taken in this section, which can easily be adapted for the continued fractions studied in [43]. Our proof uses Ford circles, and is similar to the proof of Lagrange's theorem from [53].

We begin by relating Ford circles to strong ∞ -approximants. Let $u = a/b$. Notice that if $w = c/d$, then $d \leq b$ if and only if $\text{rad}[C_u] \leq \text{rad}[C_w]$. For any real number x , let

$$R_u(x) = \frac{1}{2}|bx - a|^2.$$

Using elementary geometry, it can be shown that $R_u(x)$ is the Euclidean radius of the horocycle based at x that is externally tangent to C_u ; this is illustrated in Figure 3.21. With this terminology, we can describe a strong ∞ -approximant of a real number x as an ∞ -rational u such that for each ∞ -rational w with $\text{rad}[C_u] \leq \text{rad}[C_w]$, we have $R_u(x) \leq R_w(x)$, with equality if and only if $w = u$.

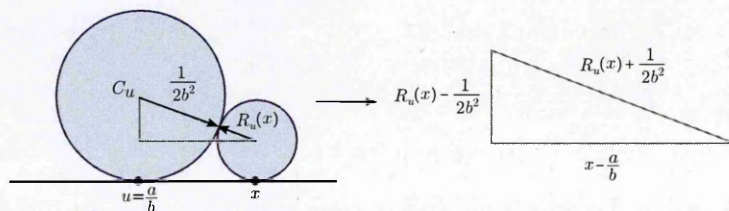


Figure 3.21: $R_u(x)$ is the Euclidean radius of the horocycle based at x that is externally tangent to C_u

It follows that we can prove [Theorem 3.20](#) by proving the following theorem.

Theorem 3.21. *Let x be an irrational number and let v be an ∞ -rational. The following are equivalent:*

1. v is a convergent of the even-integer continued fraction expansion of x ;
2. if u is an ∞ -rational such that $\text{rad}[C_u] \geq \text{rad}[C_v]$ then $R_v(x) \leq R_u(x)$ with equality if and only if $u = v$.

Before proving this theorem we look in more detail at the geometry of the collection of Ford circles. Suppose that v_i and v_{i+1} are consecutive convergents of the even-integer continued fraction expansion of x . [Lemma 3.18](#) tells us that there is a 1-rational w_i adjacent to both v_i and v_{i+1} such that both v_{i+1} and x lie between v_i and w_i . The implication of this is that the three Ford circles C_{v_i} , $C_{v_{i+1}}$ and C_{w_i} are pairwise tangent with $C_{v_{i+1}}$ lying between C_{v_i} and C_{w_i} , as illustrated in [Figure 3.22](#); the number x may lie anywhere between v_i and w_i .

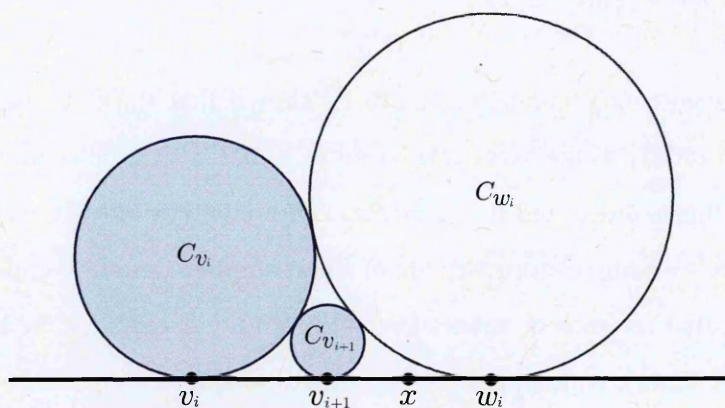


Figure 3.22: The three Ford circles C_{v_i} , $C_{v_{i+1}}$ and C_{w_i} are pairwise tangent with $C_{v_{i+1}}$ lying between C_{v_i} and C_{w_i}

To prove [Theorem 3.21](#) we require a few lemmas. The following is proved in [53, Lemma 2.2], and is clear geometrically; we omit the proof here.

Lemma 3.22. *Let C_v and C_w be tangential Ford circles. If a rational u lies strictly between v and w then C_u has smaller radius than both C_v and C_w .*

We use Lemma 3.22 to prove the following.

Lemma 3.23. *Suppose that v_i and v_{i+1} are consecutive convergents of the even-integer continued fraction expansion of x , and let w_i be the unique 1-rational adjacent to both v_i and v_{i+1} such that both v_{i+1} and x lie between v_i and w_i . If u is an ∞ -rational with $\text{rad}[C_u] \geq \text{rad}[C_{v_i}]$ then u does not lie between v_i and w_i .*

Proof. Since C_{v_i} is tangent to C_{w_i} , it follows from Lemma 3.22 that if u were to lie between v_i and w_i , then $\text{rad}[C_u] < \text{rad}[C_{v_i}]$, a contradiction. Therefore u does not lie between v_i and w_i . \square

We require another lemma.

Lemma 3.24. *Let x , v_i , v_{i+1} and w_i be as in Lemma 3.23. If u is an ∞ -rational that does not lie between v_i and w_i then $R_u(x) \geq R_{v_i}(x)$ with equality if and only if $u = v_i$.*

This lemma follows from [53, Lemma 2.3], and again is clear from the geometry of Figure 3.22, so we omit the proof.

We are now ready to prove Theorem 3.21, which in turn proves Theorem 3.20.

Proof of Theorem 3.21. Let x be an irrational number, and suppose that $v = v_i$ and v_{i+1} are consecutive convergents of the even-integer continued fraction expansion of x . Let w_i be the unique 1-rational adjacent to both v_i and v_{i+1} such that both v_{i+1} and x lie between v_i and w_i , as given by Lemma 3.18. Suppose also that u is an ∞ -rational with $\text{rad}[C_u] \geq \text{rad}[C_{v_i}]$. Then by Lemma 3.23, u does not lie between v_i and w_i , and $\text{rad}[C_u] = \text{rad}[C_{v_i}]$ if and only if $w = v_i$. It follows by Lemma 3.24 that $R_u(x) \geq R_{v_i}(x)$ with $R_u(x) = R_{v_i}(x)$ if and only if $w = v_i$.

Conversely, let x be a real number and suppose that v is an ∞ -rational with the property that if u is another ∞ -rational such that $\text{rad}[C_u] \geq \text{rad}[C_v]$ then $R_v(x) \leq R_u(x)$ with equality if and only if $u = v$. We show that v must be a convergent of an even-integer continued fraction expansion of x . Let v_1, v_2, \dots be the sequence of convergents of the even-integer continued fraction expansion of x . By Theorem 3.19 there are consecutive convergents v_i and v_{i+1} such that $\text{rad}[C_{v_{i+1}}] < \text{rad}[C_v] \leq \text{rad}[C_{v_i}]$. Let w_i be the unique 1-rational adjacent to both v_i and v_{i+1} such that both v_{i+1} and x lie between v_i and w_i . Then by Lemma 3.23, v does not lie between v_i and w_i . But then by Lemma 3.24, $R_v(x) \geq R_{v_i}(x)$ with equality if and only if $v = v_i$. Since $\text{rad}[C_v] \leq \text{rad}[C_{v_i}]$ implies that $R_v(x) \leq R_{v_i}(x)$, we have $v = v_i$, and v is a convergent of an even-integer continued fraction expansion of x . \square

3.3.3 THE HURWITZ CONSTANT

In this section we prove an analogue of Hurwitz's theorem on the approximation of irrationals by rationals for the approximation of irrational numbers by ∞ -rational numbers.

Theorem 3.25. *For any irrational number x , there are infinitely many ∞ -rationals p/q such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}. \quad (3.2)$$

Furthermore, for any $k < 1/2$ there exists an irrational number x such that there are only finitely many ∞ -rationals p/q satisfying Equation (3.2).

There is a rich literature on generalisations of Hurwitz's theorem, and we may prove Theorem 3.25 by invoking more general theorems such as those in [40]. We wish, however, to prove Theorem 3.25 using

the techniques developed within this chapter. We follow the approach taken in [51], in which a similar theorem to Theorem 3.25 is proved for the approximation of irrational numbers by 1-rationals. This proof is, in turn, based on a proof of Hurwitz’s theorem for the approximation of irrational numbers by rationals given by Ford in [13].

Our approach uses a generalisation of Ford circles. Given a reduced rational $x = a/b$, the Ford k -circle $C_x(k)$ is the horocycle based at x with Euclidean radius $\text{rad}[C_x(k)] = k/b^2$. The Ford circle $C_\infty(k)$ is the line $\text{Im}(z) = 1/(2k)$ in \mathbb{C} together with the point ∞ . When $k = \frac{1}{2}$, the Ford k -circle $C_x(k)$ is precisely the Ford circle C_x . As k decreases from $1/2$, the radius of each horocycle decreases (and the height of $C_\infty(k)$ increases) and the full collection of Ford k -circles becomes pairwise disjoint. As k increases from $1/2$, the radius of each horocycle increases (and the height of $C_\infty(k)$ decreases) and the Ford k -circles begin to overlap one another. Part of the collection of Ford k -circles for some $k < 1/2$ is shown in Figure 3.23.

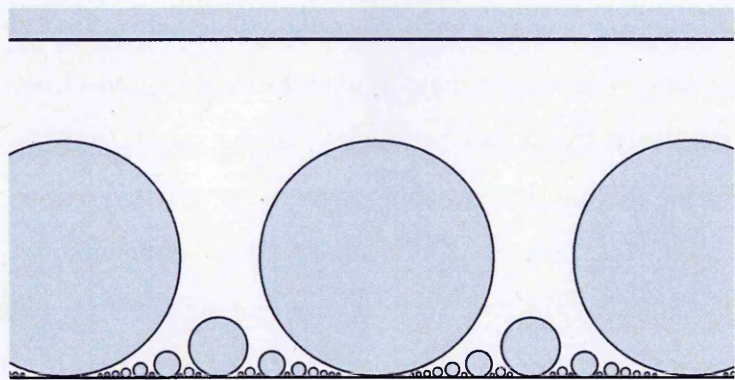


Figure 3.23: The Ford k -circles for some $k < 1/2$

The collection of Ford k -circles shares many properties with the collection of Ford circles. It is invariant under the action of the modular group Γ . We may restrict our attention to those Ford k -circles based at ∞ -rationals, and this collection is invariant under the action of the theta group Γ_∞ .

Given an irrational number x , let δ_x denote the hyperbolic geodesic joining x to ∞ . Notice that

$$\left| x - \frac{p}{q} \right| < \frac{k}{q^2}$$

if and only if δ_x intersects $C_x(k)$ (note that in general we do not consider a geodesic ℓ to intersect a Ford k -circle $C_x(k)$ if it is tangent to $C_x(k)$). We can therefore reformulate Theorem 3.25 as follows:

Theorem 3.26. *For any irrational number x , there are infinitely many ∞ -rationals y such that δ_x intersects $C_y(1/2)$. Furthermore, for any $k < 1/2$ there exists an irrational number x such that there are only finitely many ∞ -rationals y such that δ_x intersects $C_y(k)$.*

We begin by proving the first statement of the theorem.

Lemma 3.27. *For any irrational number x , there are infinitely many ∞ -rationals y such that δ_x intersects $C_y(1/2)$.*

Proof. Let x be an irrational number and let $k = 1/2$. Then any Ford k -circle is simply a Ford circle. Since the collection of Ford circles is invariant under Γ_∞ , we may assume, by applying an element of Γ_∞ if necessary, that x lies between -1 and 1 on \mathbb{R} . Consider the Ford circles that are based at the ∞ -rationals $\pm n/(n+1)$ for $n = 0, 1, \dots$. They form a chain of tangential Ford circles between -1 and 1 , as illustrated in Figure 3.24. It follows that any geodesic with an endpoint in the interval $[-1, 1]$ will intersect one of these Ford circles from this chain; let C_{p_1} be the one that δ_x intersects.

Choose an element f of Γ_∞ such that that $f(p_1) = \infty$ and $f(x)$ lies in the interval $[-1, 1]$. Then f maps C_{p_1} to C_∞ . The line δ_x is transformed to a line $\delta'_x = f(\delta_x)$ with endpoints $f(\infty)$ and $f(x)$. The portion of δ_x between C_{p_1} and x is transformed to the portion of δ'_x between C_∞ and $f(x)$. Since $f(x)$ is irrational, and lies in the interval $[-1, 1]$, the above argument shows that this portion of δ'_x intersects some Ford

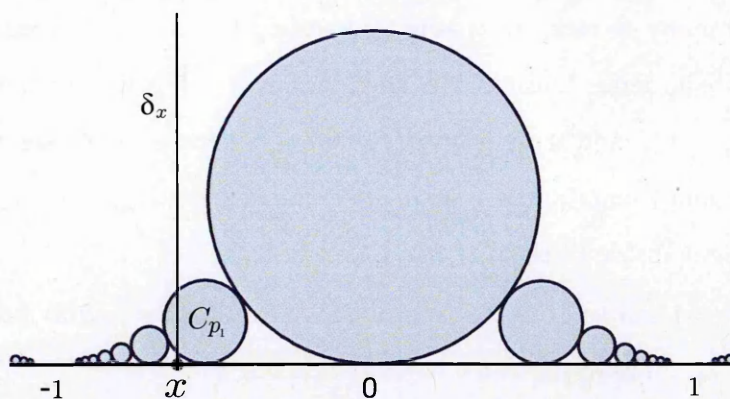


Figure 3.24: The Ford circles based at the ∞ -rationals $\pm n/(n+1)$ for $n = 0, 1, \dots$ form a chain between -1 and 1

circle C_{p_2} . Correspondingly, the line δ_x intersects both Ford circles C_{p_1} and $f^{-1}C_{p_2}$, in that order from ∞ to x . This is illustrated in Figure 3.25.

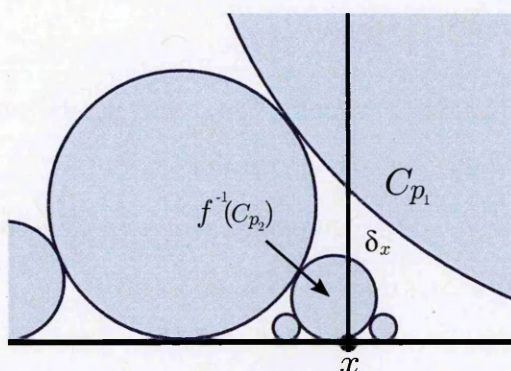


Figure 3.25: The line δ_x intersects both C_{p_1} and $f^{-1}C_{p_2}$

Repeating this argument, we build up an infinite sequence of Ford circles C_{p_i} , $i = 1, 2, \dots$, such that the Ford circles $f^{-j}(C_{p_{i+1}})$ for $j = 0, 1, \dots$ are all distinct, and all intersect δ_x . It follows that there are infinitely many ∞ -rationals y such that δ_x intersects C_y . \square

To prove the converse to Lemma 3.27, we wish to show that for any $k < 1/2$, there exists an irrational number x such that there are only finitely many ∞ -rationals y with $C_y(k)$ intersecting δ_x . To do this, we produce a sequence of irrational numbers x_n such that there are only

finitely many ∞ -rationals y with intersecting $C_y(k_n) \delta_{x_n}$, where the sequence k_n tends towards $1/2$. Then for any $k < 1/2$, we can choose $k < k_n < 1/2$, and there is an irrational x_n such that there are only finitely many ∞ -rationals y with $C_y(k)$ intersecting δ_{x_n} , since $C_y(k)$ lies strictly inside $C_y(k_n)$.

Fix $n > 1$ and let ℓ_n be the semicircle in \mathbb{H} centred at $\frac{1}{n}$ with radius $\frac{\sqrt{n^2+1}}{n}$. ℓ_n intersects \mathbb{R} at the irrational points

$$x_n = \frac{1 - \sqrt{n^2+1}}{n} \quad \text{and} \quad y_n = \frac{1 + \sqrt{n^2+1}}{n}$$

Furthermore, ℓ_n is the axis of the hyperbolic Möbius transformation

$$f_n(z) = \frac{(2n^2 - 2n + 1)z - 2n^2}{-2n^2z + 2n^2 + 2n + 1},$$

with x_n the attracting fixed point of f_n , and y_n the repelling fixed point of f_n . Note that f_n is an element of Γ_∞ .

Let $k_n = n/(2\sqrt{n^2+1})$. Elementary geometry shows that the Ford k_n -circles $A = C_{(n-1)/n}(k_n)$, $B = C_{n/(n-1)}(k_n)$, $C = C_{-n/(n+1)}(k_n)$ and $D = C_{(n+1)/n}(k_n)$ are tangent to ℓ_n . We write a and b for the points of tangency of A and B . This is illustrated in Figure 3.26.

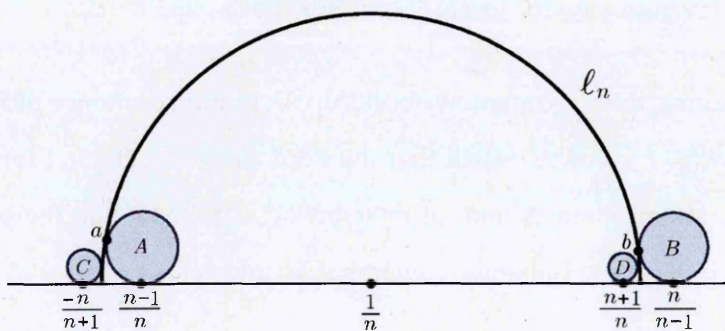


Figure 3.26: The Ford k_n -circles A , B , C and D

Lemma 3.28. *No Ford k_n -circle intersects ℓ_n between the points a and b .*

Proof. It is clear from the geometry of the collection of Ford k_n -circles that when $-\infty < x < -n/(n+1)$ or $n/(n-1) < x < \infty$, the Ford k_n -circle $C_x(k_n)$ will not intersect ℓ . When $-n/(n+1) < x < (n-1)/n$, the Ford k_n -circle $C_x(k_n)$ may intersect ℓ_n , but since $\text{Rad}[C_x(k_n)] < \text{Im}(a)/2$, any intersection point must be outside the segment of ℓ_n between a and b . Similarly, if $(n+1)/n < x < n/(n-1)$ then $C_x(k_n)$ cannot intersect ℓ_n between the points a and b . Suppose now that $x = p/q$ lies between $(n-1)/n$ and $(n+1)/n$. Then

$$\left| \frac{p}{q} - \frac{1}{n} \right| < \frac{\sqrt{n^2 + 1}}{n}. \quad (3.3)$$

Elementary geometry shows that $C_x(k_n)$ intersects ℓ_n if and only if

$$\left| \frac{p}{q} - \frac{1}{n} \right|^2 > 1 + \frac{1}{n^2} - \frac{1}{q^2},$$

which holds if and only if

$$q^2 + \frac{2pq}{n} - p^2 < 1. \quad (3.4)$$

Equation (3.3) tells us that

$$\frac{2pq}{n} - p^2 > -1.$$

Combining this with Equation (3.4), we get

$$q^2 - 1 < q^2 + \frac{2pq}{n} - p^2 < 1.$$

It follows that $q^2 < 2$, and so $q = 1$. But the only even integer x lying between $(n-1)/n$ and $(n+1)/n$ is 0, and $C_0 k_n$ is tangent to, but does not intersect, ℓ_n . Therefore no Ford k_n -circle intersects ℓ_n between a and b . \square

Corollary 3.29. *The geodesic ℓ_n is tangent to infinitely many Ford k_n -circles, but intersects none.*

Proof. Consider the segment ℓ'_n of ℓ_n lying between r_n and s_n . By Lemma 3.28, no k_n -circle based at an ∞ -rational intersects ℓ'_n . Since $f_n(c) = b$, the images of ℓ'_n under the iterates of f_n and its inverse cover ℓ_n . It follows that no Ford k_n -circle intersects ℓ_n . Clearly, the images of the Ford circles A, B, C and D under iterates of f_n and f_n^{-1} are all tangent to ℓ_n . These form an infinite family of Ford k_n -circles that are tangent to ℓ_n . \square

A part of the collection of Ford k_n -circles that are tangent to ℓ_n is shown in Figure 3.27.

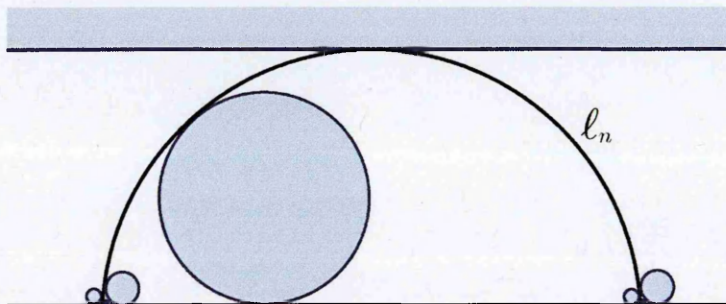


Figure 3.27: Infinitely many Ford k_n -circles based at ∞ -rationals are tangent to ℓ_n , but none intersect

We are now ready to complete the proof of Theorem 3.25.

Proof of Theorem 3.25. To prove Theorem 3.25 it is sufficient to prove Theorem 3.26. The first statement of Theorem 3.26 is given by Lemma 3.27.

To prove the second statement we wish to show that for any $k < 1/2$ there exists an irrational number x such that there are only finitely many ∞ -rationals y with δ_x intersecting $C_y(k)$.

Fix $k < 1/2$ and choose $k_n = n/(2\sqrt{n^2+1})$ such that $k < k_n$. Then by Corollary 3.29 there is a line ℓ_n with irrational endpoints x_n and y_n such that ℓ_n is tangent to infinitely many Ford k_n -circles based at ∞ -rationals, but intersects none. Consider the collection of

Ford k -circles that are based at ∞ -rationals. Since $k < k_n$, the Ford k -circle $C_\infty(k)$ is now a positive hyperbolic distance away from ℓ . The same is true for $C_0(k)$ and $C_{(n+1)/n}(k)$. Let d be the least of these three distances. Since f_n fixes ℓ_n , and the collection of Ford k -circles is invariant under f_n , it follows that all of the Ford k -circles based at points where the Ford k_n -circle was tangent to ℓ_n are at least d away from ℓ . Let A_n denote the region of \mathbb{H} bounded by the circular arcs lying distance d away from ℓ_n , as illustrated in Figure 3.28. No Ford k -circle intersects A_n .

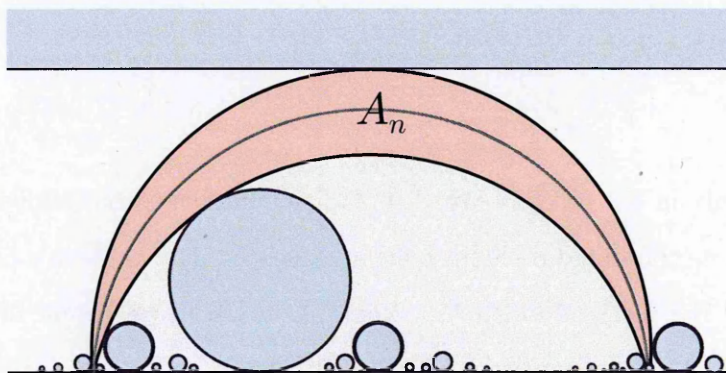


Figure 3.28: No Ford k -circles intersect the region A_n

Now consider the hyperbolic geodesic δ_{x_n} joining x_n to ∞ . The line δ_{x_n} is asymptotic to ℓ_n towards the point x_n , and hence lies within the region A_n sufficiently close to x_n . Upon leaving A_n , δ_{x_n} may only intersect finitely many Ford k -circles. It follows that x_n is an irrational number such that there are only finitely many ∞ -rationals y with δ_{x_n} intersecting $C_y(k)$. Since k was arbitrary, this completes the proof of Theorem 3.26, which in turn completes the proof of Theorem 3.25. \square

3.4 ROSEN CONTINUED FRACTIONS WHEN $\lambda > 2$

The theta group Γ_∞ , and the Hecke groups Γ_q , are only a countable collection from the larger class of groups that we also call Hecke

groups; recall that the *Hecke group* $\Gamma(\lambda)$ is the group of Möbius transformations generated by

$$\tau_\lambda(z) = z + \lambda \quad \text{and} \quad \sigma(z) = -\frac{1}{z}.$$

It was shown by Hecke [22] that $\Gamma(\lambda)$ is a Fuchsian group if and only if either $\lambda = 2\cos\left(\frac{\pi}{q}\right)$ for some integer $q \geq 3$, or if $\lambda \geq 2$. For any $\lambda > 0$, the Hecke group $\Gamma(\lambda)$ gives rise to a class of continued fractions of the form

$$b_1\lambda - \frac{1}{b_2\lambda - \frac{1}{b_3\lambda - \dots}}.$$

It is only in the cases in which $\Gamma(\lambda)$ is a Fuchsian group, however, that these continued fractions may be represented by paths in a Farey graph. The cases in which $\lambda = 2\cos\left(\frac{\pi}{q}\right)$, in which we obtain Rosen continued fractions, were studied in Chapter 2, and the special case $\lambda = 2$, in which $\Gamma(\lambda)$ is the theta group Γ_∞ and we obtain even-integer continued fractions, was the focus of this chapter. It remains to consider the cases $\lambda > 2$; here, we give a brief and informal treatment of the topic.

In the cases when $\lambda > 2$ the group $\Gamma(\lambda)$ is a Fuchsian group of the second kind, meaning that the limit set $\Lambda(\Gamma(\lambda))$ of $\Gamma(\lambda)$ is a Cantor set. The implication for the corresponding continued fractions, which we call *λ -continued fractions*, is that not every real number has a λ -continued fraction expansion; the points that can be represented by a λ -continued fraction are precisely the points that lie in $\Lambda(\Gamma(\lambda))$. The question of precisely which points lie in the set $\Lambda(\Gamma(\lambda))$ is not easy to answer, although there is some literature concerning the Hausdorff dimension of $\Lambda(\Gamma(\lambda))$ (see, for example, [1, 16]).

Just as Rosen continued fractions and even-integer continued fractions may be viewed as paths in the Farey graphs \mathcal{F}_q and \mathcal{F}_∞ respectively, λ -continued fractions may be viewed as paths in the Farey graph $\mathcal{F}(\lambda)$, which we may define in the same way as we defines the Farey graph \mathcal{F}_∞ in Section 3.1. The Farey graph $\mathcal{F}(3)$ is shown in Figure 3.29. For all $\lambda > 2$, the Farey graph $\mathcal{F}(\lambda)$ is a tree, but, unlike with \mathcal{F}_∞ , the vertices surrounding a ‘face’ of this tree do not accumulate at one point.

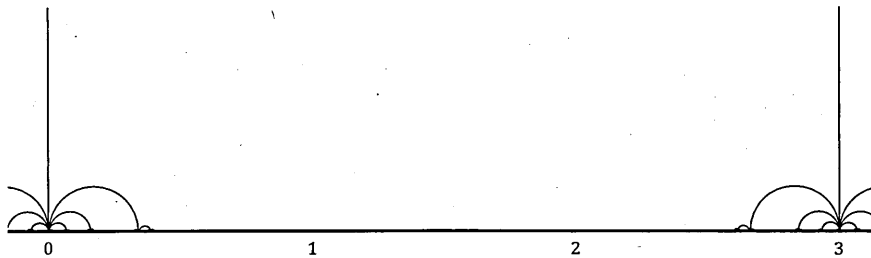


Figure 3.29: The Farey graph $\mathcal{F}(3)$

Notice that just as \mathcal{F}_∞ is a subgraph of the Farey graph \mathcal{F} , any Farey graph $\mathcal{F}(\lambda)$ with λ an integer will be a subgraph of \mathcal{F} . The Farey graph $\mathcal{F}(3)$ is shown in Figure 3.30 as a subgraph of \mathcal{F} .

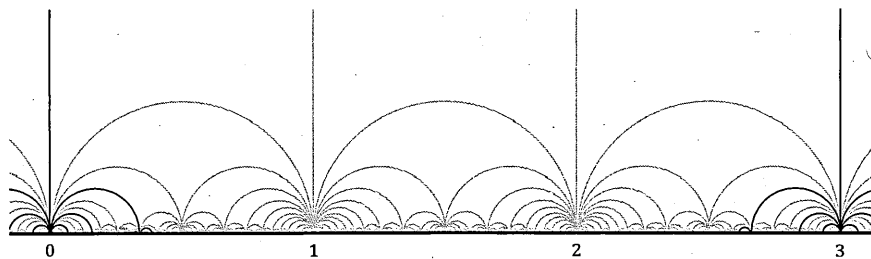


Figure 3.30: The Farey graph $\mathcal{F}(3)$ as a subgraph of the Farey graph \mathcal{F}

It follows from the structure of $\mathcal{F}(\lambda)$ that the λ -continued fraction expansion of any point in $\Lambda(\Gamma(\lambda))$ is unique, and hence geodesic. It is likely that many of the results presented in Section 3.2 and Section 3.3 will have analogues for λ -continued fractions.

FURTHER TOPICS ON CONTINUED FRACTIONS & HYPERBOLIC GEOMETRY

So far, this thesis has been concerned mainly with two classes of real continued fractions, namely Rosen continued fractions and the related even-integer continued fractions. In this chapter we look at how the techniques introduced in previous chapters can be adapted for use beyond the study of real continued fractions. We focus on two topics: complex continued fractions, and semigroups of Möbius transformations. Much of this work is in gestation, and we will not always present it with the formality of previous chapters.

Most of the work on complex continued fractions focuses on those with Gaussian integer coefficients, which we call Gaussian integer continued fractions. We show in Section 4.1.1 how Gaussian integer continued fractions can be represented by paths in a graph that occurs naturally in three-dimensional hyperbolic geometry. This representation allows us to use techniques similar to those used in previous chapters to study Gaussian integer continued fractions. A basic result about the convergence of infinite Gaussian integer continued fractions, which generalises Theorem 2.39, is proved, and geodesic Gaussian integer continued fractions are discussed. Section 4.2 looks at a class of continued fractions, which we call Bianchi continued fractions, whose coefficients lie in the set $\mathbb{Z}[\sqrt{2}i]$. We take an entirely different approach in this section to that in Section 4.1, developing a cutting sequence that allows us to produce a new type of continued fraction expansion of a complex number.

In Section 4.3 we generalise classical regular continued fractions in an entirely new direction. We see that regular continued fractions can be viewed as sequences of elements of a semigroup of Möbius transformations. Given a two-generator semigroup of Möbius transformations S that satisfies certain properties, we show how we can define a more general notion of a regular continued fraction, which we call an S -continued fraction. We discuss potential areas of research into S -continued fractions.

4.1 GAUSSIAN INTEGER CONTINUED FRACTIONS

A *Gaussian integer continued fraction* is a continued fraction of the form

$$b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}},$$

where each b_i is a Gaussian integer, that is, each b_i lies in the set

$$\mathbb{Z}[i] = \{z = a + bi \in \mathbb{C} \mid x, y \in \mathbb{Z}\}.$$

We denote an infinite Gaussian integer continued fraction by $[b_1, b_2, \dots]$, and a finite Gaussian integer continued fraction by $[b_1, \dots, b_n]$.

The study of Gaussian integer continued fractions dates back at least as far as the work of the brothers Adolf and Julius Hurwitz (see [37]). Since then, a reasonably large literature on Gaussian integer continued fractions has emerged. Many of these works make use of geometry in their arguments, such as Ford [14, 15], LeVeque [33], Schmidt [44, 45] and Vulakh [61]. Within these works, many different algorithms for producing a Gaussian integer continued fraction expansion of a complex number are presented, each of which can

be considered a generalisation of the regular continued fraction algorithm for real numbers. The work of Dani and Nogueira [11] appears to be the first to study Gaussian integer continued fractions more broadly, using a definition of Gaussian integer continued fractions that encompasses all previously studied classes of Gaussian integer continued fraction. Yet no attempt has been made to develop a theory of the class of *all* Gaussian integer continued fractions. The aim of this section is to outline how such a theory may be built up using the techniques from hyperbolic geometry that have been discussed so far in this thesis.

We begin this section by showing that Gaussian integer continued fractions can be represented by paths in a graph embedded in three-dimensional hyperbolic space that is a natural analogue of the Farey graphs introduced in previous chapters of this thesis. We use this perspective to obtain a new result on the convergence of infinite Gaussian integer continued fractions. We then propose several areas for further research in which this technique is likely to yield results, and discuss how our approach may give insight into the work of Dani and Nogueira.

4.1.1 GAUSSIAN INTEGER CONTINUED FRACTIONS AND HYPERBOLIC GEOMETRY

The *Picard group* Γ_P is the group of Möbius transformations generated by

$$\tau_1(z) = z + 1, \quad \tau_i(z) = z + i, \quad \sigma(z) = -\frac{1}{z} \quad \text{and} \quad \sigma'(z) = \frac{1}{z} = \frac{i}{iz}.$$

The group Γ_P is a natural analogue of the modular group Γ , and is hence sometimes referred to as the Picard modular group. It is

isomorphic to the group $\text{PSL}(2, \mathbb{Z}[i])$, and consists precisely of those Möbius transformations that can be written

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c , and d are Gaussian integers, and $ad - bc = 1$. Writing $\rho = \sigma\tau_1$ and $\rho' = \sigma'\tau_1^{-1}$, we may write down a presentation of Γ_P as

$$\begin{aligned} \Gamma_P = \langle \sigma, \sigma', \rho, \rho' \mid \sigma^2 = \sigma'^2 = \rho^3 = \rho'^3 = (\rho\sigma')^2 \\ = (\rho'\sigma)^2 = (\rho\rho')^2 = (\sigma\sigma')^2 = I \rangle. \end{aligned}$$

It is possible to write down a presentation of Γ_P with as few as two generators – see, for example, [5] – but the generators given in the definition of the Picard group above allow us to easily see the connection between Gaussian integer continued fractions and the Picard group.

Consider a finite Gaussian integer continued fraction $[b_1, \dots, b_n]$. Write $b_k = x_k + y_k i$ for each $k = 1, \dots, n$. As usual, we define the sequence of Möbius transformations

$$f_k(z) = b_k - \frac{1}{z},$$

for $k = 1, 2, \dots, n$, that is, $f_k = \tau_1^{x_k} \tau_i^{y_k} \sigma$ using the generators τ_1, τ_i and σ of Γ_P . Then the convergents of $[b_1, \dots, b_n]$ are given by

$$v_k = f_1 \cdots f_k(\infty),$$

for $k = 1, \dots, n$, and a Gaussian integer continued fraction can be viewed as a sequence $F_k = f_1 \cdots f_k$ of Möbius transformations belonging to the Picard group. Similarly, an infinite Gaussian integer continued fraction can be viewed as an infinite sequence of Möbius transformations belonging to the Picard group.

The action of the Picard group Γ_P on the upper half-space \mathbb{H}^3 gives rise to a representation of Gaussian integer continued fractions using hyperbolic geometry: the group Γ_P is a Kleinian group, so Γ_P acts properly discontinuously on \mathbb{H}^3 . The hyperbolic polyhedron

$$D = \{z + tj \in \mathbb{H}^3 \mid 0 < \operatorname{Re}(z) < 1, 0 < \operatorname{Im}(z) < 1/2, \\ |z + tj| > 1, |z - 1 + tj| > 1\},$$

is a fundamental domain for Γ_P with side-pairing transformations τ_1 , $\tau_1\sigma$, $\tau_1\sigma\sigma'$ and $\tau_1\tau_1\sigma\sigma'$. The polyhedron D is shown in Figure 4.1 (a).

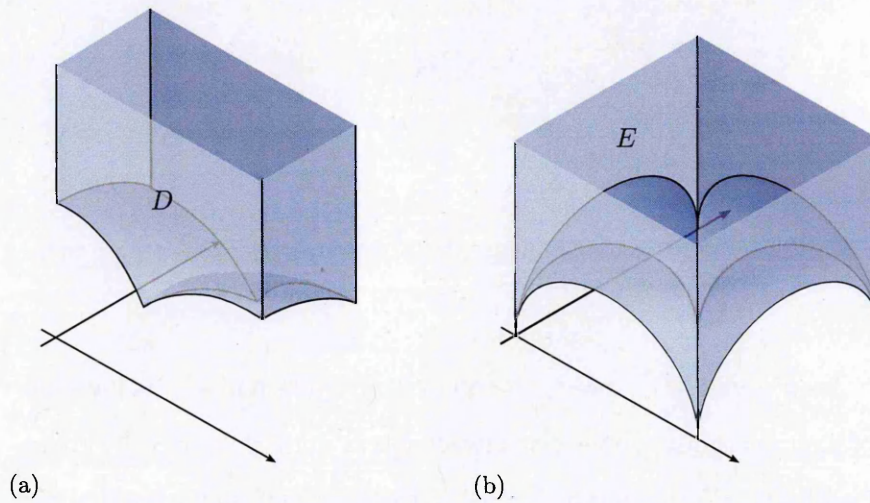


Figure 4.1: (a) A fundamental domain for Γ_P (b) A fundamental domain for Θ_P

Let v be the vertex $1/2 + i/2 + j/\sqrt{2}$ of D . The stabiliser of the point v under the group Γ_P is the group $\operatorname{Stab}_{\Gamma_P}(v)$ generated by the maps $\tau_1\sigma(z) = 1 - 1/z$ and $\tau_1\sigma'(z) = i + 1/z$; it consists of 12 elements and is isomorphic to the alternating group A_4 (see [17]). The union of the images of D under the elements of $\operatorname{Stab}_{\Gamma_P}(v)$ is a closed ideal octahedron E with vertices $\infty, 0, 1, 1 + i, i$ and $1 - i$. The region E is shown in Figure 4.1 (b). It is shown by Vulakh [61] that E is the fundamental domain for a normal subgroup Θ_P of Γ_P with $\Gamma_P/\Theta_P \cong \operatorname{Stab}_{\Gamma_P}(v)$. The images of E under Θ_P tessellate \mathbb{H}^3 by ideal hyperbolic octahe-

dra. The skeleton of this tessellation is a connected graph, which we call the Farey graph \mathcal{F}_P . The vertices of \mathcal{F}_P are the ideal vertices of the tessellation; they belong to the ideal boundary \hat{C} of \mathbb{H}^3 . The edges of \mathcal{F}_P are the edges of the ideal octahedra in the tessellation, and the faces of \mathcal{F}_P are the faces of the ideal octahedra. Part of \mathcal{F}_P is shown in Figure 2.2.

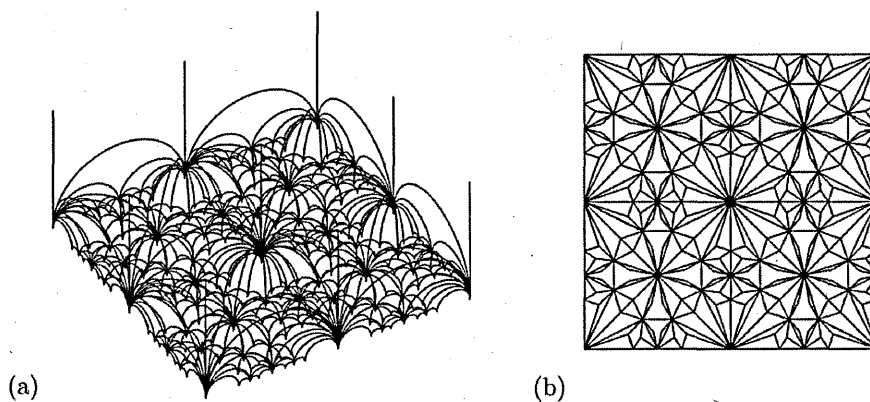


Figure 4.2: (a) Part of the Farey graph \mathcal{F}_P (b) The projection of part of the Farey graph \mathcal{F}_P onto \mathbb{C}

As in previous chapters, we can define Γ_P alternatively as the orbit of the hyperbolic geodesic δ between 0 and ∞ in \mathbb{H}^3 under the group Γ_P . The transformation $\sigma(z) = -1/z$ maps ∞ to 0, so we see that the set of vertices of \mathcal{F}_P is the orbit of ∞ under Γ_P . We therefore call the vertices of \mathcal{F}_P *∞ -rationals*. The set of ∞ -rationals is precisely the set of reduced Gaussian rationals, which is the set of all complex numbers of the form $x + yi$ where x and y are reduced rationals, together with the point ∞ , which we identify with $1/0$.

As in Section 2.1 and Section 3.1, this alternative description of \mathcal{F}_P allows us to determine the neighbours of ∞ in \mathcal{F}_P ; we find that they are precisely the Gaussian integers. It also allows us to determine the automorphism group of \mathcal{F}_P . It follows from the fact that \mathcal{F}_P is the orbit of δ under Γ_P that each element of Γ_P induces an automorphism of \mathcal{F}_P . The two maps $\kappa(z) = -\bar{z}$ and $\mu(z) = iz$ also induce automor-

phisms of \mathcal{F}_p . The group generated by Γ_p , κ and μ is the *extended Picard group*, $\tilde{\Gamma}_p$, and is in fact the full group of automorphisms of \mathcal{F}_p .

Theorem 4.1. *The extended Picard group $\tilde{\Gamma}_p$ is the full group of automorphisms of \mathcal{F}_p .*

We omit the proof of this theorem, which is similar to the proof of Theorem 2.1 that is sketched in Section 2.1. The automorphisms of $\tilde{\Gamma}_p$ not only preserve incidence between vertices and edges of \mathcal{F}_p , in fact they also preserve incidence between vertices, edges, faces and octahedra.

We are now in a position to state the theorem that explains the correspondence between Gaussian integer continued fractions and paths in Farey graphs. For now, we will concentrate on finite Gaussian integer continued fractions and finite paths.

Theorem 4.2. *Let v be a vertex of \mathcal{F}_p other than ∞ . Then the vertices v_1, \dots, v_n of \mathcal{F}_p , with $v_n = v$, are the consecutive convergents of some Gaussian integer continued fraction expansion of v if and only if $\langle \infty, v_1, \dots, v_n \rangle$ is a path in \mathcal{F}_p from ∞ to v .*

Again, we omit the proof of this theorem, as it follows similarly to the proof of Theorem 2.2. We call $\langle \infty, v_1, \dots, v_n \rangle$ the *path of convergents* of $[b_1, \dots, b_n]$. In Section 2.1.3 a simple method for moving between a Rosen continued fraction and its corresponding path of convergents in the Farey graph \mathcal{F}_q was presented; a similar method for moving between an even-integer continued fraction and its path of convergents in \mathcal{F}_∞ was also presented in Section 3.1.3. The situation is considerably more complicated when moving between a Gaussian integer continued fraction and its path of convergents as the collection of neighbours of any given vertex does not have a natural order; we omit any further discussion of this.

We end this section by briefly discussing infinite Gaussian integer continued fractions and infinite paths, which will be the focus of Sec-

tion 4.1.2. An *infinite path* in \mathcal{F}_P is a sequence of vertices v_0, v_1, \dots such that $v_{i-1} \sim v_i$ for $i = 1, 2, \dots$. It follows from Theorem 4.2 that the convergents of an infinite Gaussian integer continued fraction form an infinite path $\langle \infty, v_1, v_2, \dots \rangle$ in \mathcal{F}_P , and conversely each infinite path of this type is comprised of the convergents of an infinite Gaussian integer continued fraction.

4.1.2 CONVERGENCE OF INFINITE GAUSSIAN INTEGER CONTINUED FRACTIONS

It is shown in Theorem 2.39 that an infinite integer continued fraction converges if its sequence of convergents does not contain infinitely many terms that are equal. Conversely, if an integer continued fraction converges to some value v , then it can only have infinitely many terms that are equal to some rational number w if $w = v$.

Theorem 4.3. *If the sequence of convergents of an infinite Gaussian integer continued fraction does not contain infinitely many terms that are equal, and has only finitely many distinct accumulation points, then that Gaussian integer continued fraction converges.*

As with Theorem 2.39, there is an obvious converse to Theorem 4.3: if an infinite Gaussian integer continued fraction converges to some value v , then v is the unique accumulation point of its sequence of convergents, which may only have infinitely many terms equal to some complex rational w if $w = v$.

The proof of Theorem 2.39 for integer continued fractions relies on the property of the Farey graph \mathcal{F} that removing any two vertices separates \mathcal{F} into two components. This is not the case in \mathcal{F}_P : it can be shown that given a finite collection of vertices v_1, \dots, v_n of \mathcal{F}_P , the graph obtained from \mathcal{F}_P by removing the vertices v_1, \dots, v_n is connected.

We define a *Farey section* to be an image of $\hat{\mathbb{R}}$ under an element of the extended Picard group $\tilde{\Gamma}_P$. A Farey section is either a Euclidean circle in $\hat{\mathbb{C}}$ or, if the Farey section contains ∞ , a Euclidean straight line in $\hat{\mathbb{C}}$. The full collection of Farey sections that contain the point ∞ are the lines $\text{Im}(z) = c$ and $\text{Re}(z) = c$ for all integers c . It follows that at any vertex of \mathcal{F}_P , infinitely many Farey sections meet, forming a degenerate family of coaxial circles. Such a family of Farey sections is illustrated in Figure 4.3.

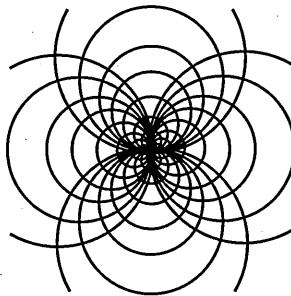


Figure 4.3: A family of Farey sections that are mutually tangent to a vertex of \mathcal{F}

Farey sections are so called because the subgraph of \mathcal{F}_P with vertices on $\hat{\mathbb{R}}$ is an embedding of the Farey graph \mathcal{F} into the half-plane $\{x + yi + tj \mid y = 0\}$; edges of this Farey graph are edges of \mathcal{F}_P , and the faces are faces of \mathcal{F}_P .

To prove Theorem 4.3 we use the following two lemmas, which are analogues of the two lemmas Lemma 2.40 and Lemma 2.5 that are used in the proof of Theorem 2.39.

Lemma 4.4. *Suppose that x and y are distinct elements of $\hat{\mathbb{C}}$ that are not vertices of a common octahedron of \mathcal{F}_P . Then there is a Farey section S such that x and y lie in distinct components of $\hat{\mathbb{C}} \setminus S$.*

Proof. Let x and y be distinct elements of $\hat{\mathbb{C}}$ that are not vertices of a common octahedron of \mathcal{F}_P . Consider the hyperbolic line δ from x to y in \mathbb{H}^3 . The line δ cannot lie within one octahedron and hence must intersect either an edge or a face of \mathcal{F}_P . Suppose that δ intersects

an edge of \mathcal{F}_P , which we may assume, after applying an element of Γ_P , is the edge between ∞ and 0. Then at least one of $\hat{\mathbb{R}}$ and $\hat{\mathbb{R}}i$ is a Farey section \mathcal{S} such that x and y lie in distinct components of $\hat{\mathbb{C}} \setminus \mathcal{S}$. Suppose instead that δ intersects a face of \mathcal{F}_P . After applying an element of Γ_P we may assume that this face has both ∞ and 1 as vertices, along with either ± 1 or $\pm i$. Again, at least one of $\hat{\mathbb{R}}$ and $\hat{\mathbb{R}}i$ is then a Farey section \mathcal{S} such that x and y lie in distinct components of $\hat{\mathbb{C}} \setminus \mathcal{S}$. \square

Lemma 4.5. *Let \mathcal{S} be a Farey section. Suppose that γ is a path in \mathcal{F}_P that starts at a vertex in one component of $\hat{\mathbb{C}} \setminus \mathcal{S}$ and finishes at a vertex in the other component of $\hat{\mathbb{C}} \setminus \mathcal{S}$. Then γ passes through a vertex lying on \mathcal{S} .*

Proof. We may assume, by applying an element of Γ_P if necessary, that $\mathcal{S} = \hat{\mathbb{R}}$ or $\mathcal{S} = \hat{\mathbb{R}}i$. We will suppose that $\mathcal{S} = \hat{\mathbb{R}}$; the proof when $\mathcal{S} = \hat{\mathbb{R}}i$ is similar. If the lemma is false, there is an edge of γ with endpoints in each of the components of $\hat{\mathbb{C}} \setminus \hat{\mathbb{R}}$. This edge intersects the half-plane $\{x + yi + tj \mid y = 0\}$, which is a contradiction because this plane consists entirely of edges and faces of \mathcal{F}_P . \square

Proof of Theorem 4.3. We prove the contrapositive of Theorem 4.3. Suppose that the sequence of convergents v_1, v_2, \dots of an infinite Gaussian integer continued fraction diverges. Then this sequence has two distinct accumulation points, say x and y . We assume first that x and y lie on a common octahedron of \mathcal{F}_P . After applying an element of Γ_P , we can assume that $x = \infty$ and y equals either 0, 1, i or $(1+i)/2$. If the sequence v_1, v_2, \dots has infinitely many terms equal to either ∞ or y then the result follows immediately. Else we may assume, by removing a finite number of terms of the sequence v_1, v_2, \dots if necessary, that no v_i is equal ∞ or y . Consider the Euclidean squares S_k with vertices $\pm k \pm ki$ for $k = 2, 3, \dots$. Since the sequence v_1, v_2, \dots accumulates at both ∞ and y , but has no term equal to ∞ , it con-

tains infinitely many terms in \mathbb{C} lying within S_k , and infinitely many terms lying outside S_k . Since the edges of S_k belong to Farey sections, it follows from Lemma 4.5 that for each k , the square S_k contains a complex number z_k such that the sequence v_1, v_2, \dots either has infinitely many terms equal to z_k , or has an accumulation point at z_k .

We assume now that x and y do not lie on a common octahedron of \mathcal{F}_P . By Lemma 4.4 there is a Farey section S_1 such that x and y lie in distinct components of $\hat{\mathbb{R}} \setminus S_1$. Since both x and y are accumulation points of the sequence v_1, v_2, \dots , it follows from Lemma 4.5 that sequence contains infinitely many terms that are vertices of \mathcal{F}_P lying on S_1 . Then the sequence v_1, v_2, \dots either has infinitely many terms that are equal, in which case we are done, or has an accumulation point z_1 lying on S_1 . If z_1 and x lie on a common octahedron of \mathcal{F}_P then the result follows as above, otherwise by Lemma 4.4 there is a Farey section S_2 such that x and z_1 lie in distinct components of $\hat{\mathbb{R}} \setminus S_2$. Since both x and z_1 are accumulation points of the sequence v_1, v_2, \dots , it follows from Lemma 4.5 that sequence contains infinitely many terms that are vertices of \mathcal{F}_P lying on S_2 . Then the sequence v_1, v_2, \dots either has infinitely many terms that are equal, in which case we are done, or has an accumulation point z_2 lying on S_2 . We may repeat this argument to show that the sequence v_1, v_2, \dots either has infinitely many terms that are equal or has infinitely many accumulation points.

This completes the proof of Theorem 4.3. \square

We cannot omit from the hypothesis of Theorem 4.3 the condition that the sequence of convergents of an infinite Gaussian integer continued fraction has only finitely many distinct accumulation points, as is the case with integer continued fractions.

Lemma 4.6. *There exists an infinite Gaussian integer continued fraction whose convergents are all distinct, but which does not converge.*

Proof. We construct a simple path in \mathcal{F}_P with two accumulation points. Let x and y be two complex numbers, and let x_1, x_2, \dots and y_1, y_2, \dots be two sequences of ∞ -rationals such that $x_k \rightarrow x$ and $y_k \rightarrow y$ and $k \rightarrow \infty$. Let P_1 be a path from ∞ to x_1 . Since removing finitely many vertices does not disconnect \mathcal{F}_P , we may find a path P_2 from x_1 to y_1 such that P_1 and P_2 intersect only at x_1 . Continuing in this fashion, we may construct a simple path from ∞ that passes through each vertex x_i and y_i , and hence converges to both x and y . The Gaussian integer continued fraction corresponding to this path of convergents has convergents that are all distinct, but does not converge. \square

We end this section by illustrating, in Figure 4.4, part of a path $\langle v_1, v_2, \dots \rangle$ (viewed as a projection onto \mathbb{C}) that diverges and yet no two terms in the sequence v_1, v_2, \dots are equal. The sequence accumulates at every real number in the interval $[0, 1]$.

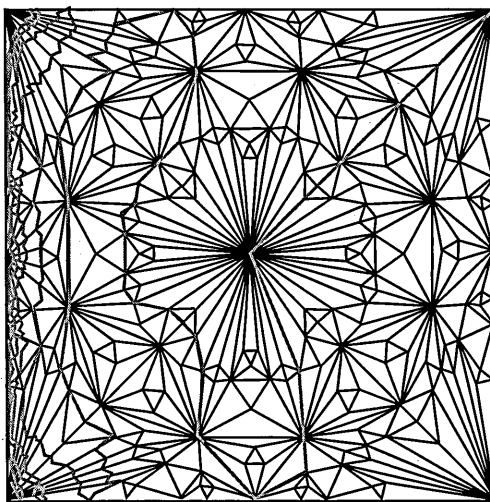


Figure 4.4: A divergent path whose vertices are all distinct

4.1.3 AREAS FOR FURTHER RESEARCH

In this section we look at directions in which the approach to the study of Gaussian integer continued fractions outlined in this chapter might lead.

Geodesic Gaussian integer continued fractions

Each vertex x of \mathcal{F}_P has infinitely many Gaussian integer continued fraction expansions. Numerous examples of algorithms for obtaining a Gaussian integer continued fraction expansion of a complex number may be found in the literature, such as the nearest-Gaussian integer algorithm introduced by Adolf Hurwitz, or more recent algorithms introduced by LeVeque [33], Schmidt [44, 45], Hayward [21] and Dani and Nogueira [11]. As in Chapter 2, we may turn our attention to geodesic Gaussian integer continued fractions.

We say that a finite Gaussian integer continued fraction expansion $[b_1, \dots, b_n]$ of a vertex x of \mathcal{F}_P is *geodesic* if every other Gaussian integer continued fraction expansion of x has at least n terms. A geodesic Gaussian integer continued fraction is so called because its corresponding path in \mathcal{F}_P is geodesic, that is, a path with the least number of edges. We say that an infinite Gaussian integer continued fraction expansion $[b_1, b_2, \dots]$ of a complex number x is *geodesic* if $[b_1, \dots, b_k]$ is a geodesic Gaussian integer continued fraction for all $k = 1, 2, \dots$

Key to the study of geodesic Rosen continued fractions is concept of the q -chain between two vertices u and v of the Farey graph \mathcal{F}_q , which was introduced in Section 2.2. The q -chain between u and v is a sequence of faces of \mathcal{F}_q in which every geodesic path between u and v lies. We conjecture that between any two vertices u and v of \mathcal{F}_P , an analogous chain of octahedra exists in which every geodesic

path from u to v lies. We discuss informally how such a chain may be produced.

In the following, we let p denote the vertical projection $p : \mathbb{H}^3 \rightarrow \mathbb{C}$ onto \mathbb{C} given by $p(z + tj) = z$. Let u and v be two distinct, non-adjacent vertices of \mathcal{F}_p . We may assume, after applying some element of Γ_p , that u is the point ∞ . If v lies on an octahedron that is incident to ∞ , then we call this octahedron P_1 and we are done. If not, then there is some octahedron, P_1 , such that v lies within the closed region $p(P_1)$ of \mathbb{C} . There may be two such octahedra, in which case we make a choice based on some predetermined rule. Consider the faces of P_1 . We choose one of these faces F_1 so that v lies within the closed region $p(F_1)$ of \mathbb{C} ; again, in an ambiguous case, we make a choice based on some predetermined rule. There is precisely one octahedron of \mathcal{F}_p other than P_1 that is incident to F_1 , and we call this P_2 . If v is a vertex of P_2 then we are done. Else there is some face F_2 of P_2 such that v lies within the closed region $p(F_2)$ of \mathbb{C} ; again, in an ambiguous case, we make a choice based on some predetermined rule. Continuing in this fashion, we produce a chain of octahedra P_1, \dots, P_n such that ∞ is a vertex of P_1 ; it remains to be shown that eventually the procedure will terminate, and so v is a vertex of P_n for some n . An example of such a sequence of faces F_1, \dots, F_4 , viewed as a projection onto \mathbb{C} , is shown in Figure 4.5.

The projections of the faces F_1, \dots, F_n are closely related to the chains of *Farey triangles* introduced by Schmidt [45] to study Diophantine approximation of complex numbers. It is likely that studying the chains of octahedra described above would illuminate the work of Schmidt.

We make the following conjecture.

Conjecture. *Any geodesic path from ∞ to a Gaussian rational number v is contained in the chain of octahedra between ∞ and v .*

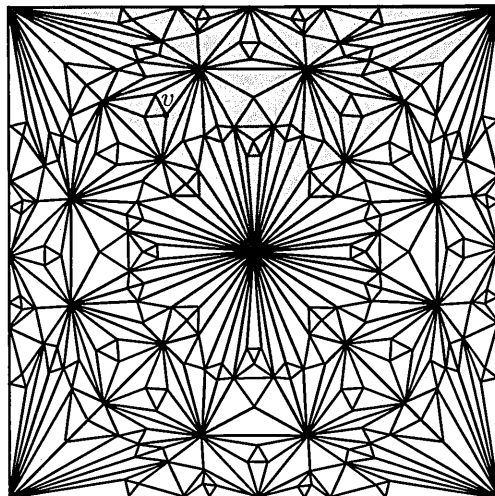


Figure 4.5: A chain of faces F_1, \dots, F_4 of octahedra forming a chain from ∞ to v , viewed as projections onto \mathbb{C}

Proving this conjecture would allow us to develop a theory of geodesic Gaussian integer continued fractions analogous to the theory of geodesic Rosen continued fractions presented in Chapter 2. We would be able to obtain bounds on the number of geodesic Gaussian integer continued fractions of a vertex x of \mathcal{F}_P , and give necessary and sufficient conditions on the coefficients b_i for a Gaussian integer continued fraction $[b_1, b_2, \dots]$ to be geodesic. It would then, most probably, follow from Theorem 4.3 that all infinite geodesic Gaussian integer continued fractions converge.

The approach of Dani and Nogueira

Many of the number-theoretic properties of Gaussian integer continued fractions have been well studied, including the approximation of complex irrational numbers by Gaussian rationals (see, for example [14], [45], [60], [61]). Much of this research, however, restricts to Gaussian integer continued fractions given by specific algorithms. The work of Dani and Nogueira [11], however, studies Gaussian integer continued fractions in more generality than in previous works. In this section we will briefly outline the work of Dani and Nogueira,

discussing how the ideas might be reformulated in terms of the Farey graph \mathcal{F}_p .

In [11], the authors introduce the notion of a *choice function*, which is a function $f : \mathbb{C} \rightarrow \mathbb{Z}[i]$ such that $f(z)$ lies in the closed ball $\overline{B}(z, 1)$ for all z in \mathbb{C} . A choice function can be used to produce a Gaussian integer continued fraction algorithm in the following way: given $z \in \mathbb{C}$, we associate two sequences z_0, z_1, \dots and b_0, b_1, \dots with

$$z_0 = z, \quad b_0 = f(z_0),$$

and for each $n \geq 0$

$$z_{n+1} = \frac{1}{z_n - b_n} \quad \text{and} \quad b_{n+1} = f(z_{n+1}).$$

Then for each $n > 0$, $z_n = a_n + 1/z_{n+1}$, and

$$b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \cdots + \frac{1}{z_n}}}.$$

Therefore a choice function gives rise to a Gaussian integer continued fraction algorithm. Conversely, many of the Gaussian integer continued fraction algorithms that can be found in the literature may be described by a choice function.

The authors of [11] introduce a still more general way of obtaining a Gaussian integer continued fraction expansion of a complex number. An *iteration sequence* for a complex number z is a sequence of complex numbers z_0, z_1, \dots such that

$$z_0 = z, \quad z_0 - \frac{1}{z_1} \in \mathbb{Z}[i],$$

and for all $n > 0$,

$$z_n - \frac{1}{z_{n+1}} \in \mathbb{Z}[i] \setminus \{0\}.$$

Writing $b_n = z_n - 1/z_{n+1}$ for each $n \geq 0$, we obtain a continued fraction as above. We can obtain an iteration sequence from any Gaussian integer continued fraction algorithm or choice function, but there are iteration sequences that cannot be obtained from any algorithm or choice function. The class of continued fraction expansions obtained through iteration sequences is, therefore, relatively general.

In [11], results are obtained that give conditions for the continued fractions obtained through choice functions or iteration sequences to converge. An analogue of Lagrange's theorem (see Section 3.2.5) for iteration sequences is given too.

It is likely that both choice functions and iteration sequences have a geometric interpretation within the Farey graph \mathcal{F}_P ; perhaps, for example, the corresponding continued fractions will always lie within the chain of octahedra in \mathcal{F}_P described above. If this is the case, then the results of Dani and Nogueira in [11] may be illuminated by studying the paths within these chains. In particular, it might be possible to better understand the conditions for convergence given in [11]; this could potentially lead to new conditions on the coefficients b_i for the Gaussian integer continued fraction $[b_1, b_2, \dots]$ to converge.

4.2 BIANCHI CONTINUED FRACTIONS

We define a *Bianchi continued fraction* to be a continued fraction of the form

$$b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}},$$

where each b_i lies in the set

$$\mathbb{Z}[\sqrt{2}i] = \{z = x + y\sqrt{2}i \in \mathbb{C} \mid x, y \in \mathbb{Z}\}.$$

As usual, we denote an infinite Bianchi continued fraction by $[b_1, b_2, \dots]$, and a finite Bianchi continued fraction by $[b_1, \dots, b_n]$. Bianchi continued fractions are so called because they arise in the study of a Kleinian group called the Bianchi group, much in the same way that Gaussian integer continued fractions arise in the study of the Picard group. Since the set $\mathbb{Z}[\sqrt{2}i]$ is a natural analogue of the set $\mathbb{Z}[i]$, Bianchi continued fractions are a natural analogue of Gaussian integer continued fractions. There has, however, been little research into Bianchi continued fraction, aside from works by Schmidt [44, 46] and Vulakh [59, 62] that focus on the approximation of complex numbers by elements of the imaginary quadratic field $\mathbb{Q}[\sqrt{2}i] = \{x + y\sqrt{2}i \in \mathbb{C} \mid x, y \in \mathbb{Q}\}$.

In this section we take a thoroughly different approach to that used in Section 4.1 to study Gaussian integer continued fractions. In particular, we adapt a technique used by Moeckel [35] and Series [52] (see Section 1.2.4) in order to associate to each geodesic in \mathbb{H}^3 a Bianchi continued fraction. This leads to a new Bianchi continued fraction expansion of a complex number.

4.2.1 BIANCHI CONTINUED FRACTIONS AND HYPERBOLIC GEOMETRY

In this section we introduce the Farey graph \mathcal{F}_B that forms the basis of our connection between geodesics in \mathbb{H}^3 and Bianchi continued fractions. This graph is a direct analogue of the Farey graph \mathcal{F}_P that was introduced in Section 4.1 and used to study Gaussian integer continued fractions. Because of the similarity between \mathcal{F}_B and \mathcal{F}_P , we omit many of the details in our discussion of the construction of \mathcal{F}_B .

Let $\omega = \sqrt{2}i$. The *Bianchi group* Γ_B is the group of Möbius transformations generated by

$$\tau_1(z) = z + 1, \quad \tau_\omega(z) = z + \omega, \quad \text{and} \quad \sigma(z) = -\frac{1}{z}.$$

The Bianchi group was introduced by Luigi Bianchi [4] as a natural example of a Kleinian group of the first kind. It consists of precisely those Möbius transformations of the form

$$f(z) = \frac{az + b}{cz + d}, \quad (4.1)$$

where a, b, c , and d lie in $\mathbb{Z}[\omega]$ and $ad - bc = 1$, and is hence isomorphic to $\text{PSL}(2, \mathbb{Z}[\omega])$. As in Section 4.1, a Bianchi continued fraction $[b_1, \dots, b_n]$ can be viewed as a sequence $F_k = f_1 \cdots f_k$ of Möbius transformations belonging to the Bianchi group. Similarly, an infinite Bianchi continued fraction can be viewed as an infinite sequence of Möbius transformations belonging to the Bianchi group.

To produce the graph \mathcal{F}_B we use the extended Bianchi group rather than the Bianchi group. The *extended Bianchi group* $\tilde{\Gamma}_B$ is the group of Möbius transformations generated by Γ_B and the map $\sigma'(z) = \frac{1}{z}$. It consists of all Möbius transformations of the form given in Equation

(4.1) where a, b, c , and d lie in $\mathbb{Z}[\omega]$ and $|ad - bc| = 1$, and it contains Γ_B as an index two normal subgroup. The hyperbolic polyhedron

$$D = \{z + tj \in \mathbb{H}^3 \mid 0 < \operatorname{Re}(z) < 1, 0 < \operatorname{Im}(z) < \omega/2, \\ |z + tj| > 1, |z - 1 + tj| > 1\},$$

is a fundamental domain for $\tilde{\Gamma}_B$ with side-pairing transformations τ_1 , $\tau_1\sigma$, $\tau_1\sigma\sigma'$ and $\tau_\omega\tau_1\sigma\sigma'$. The polyhedron D is shown in Figure 4.6 (a).

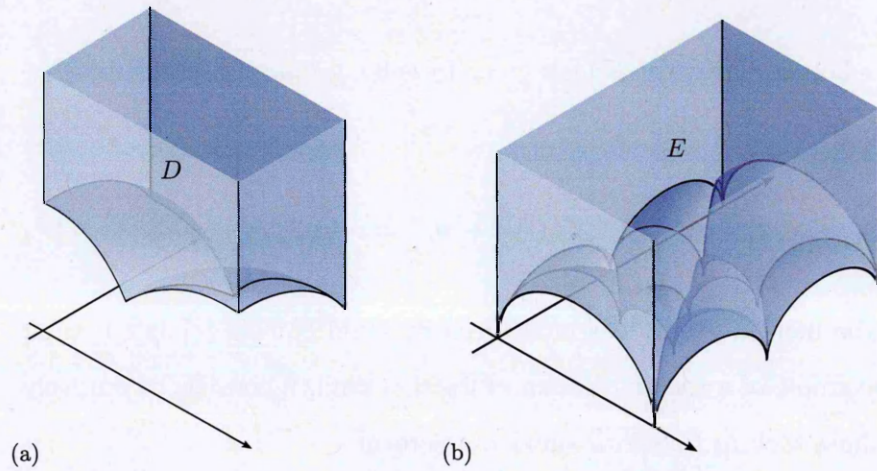


Figure 4.6: (a) A fundamental domain for $\tilde{\Gamma}_B$ (b) A fundamental domain for Θ_B

Let v be the vertex $1/2 + \omega i/2 + j/2$ of D . The union of the images of D under the elements of the stabiliser of the point v under $\tilde{\Gamma}_B$ is the closed ideal cuboctahedron E with vertices $\infty, 0, 1, 1 + \omega, \omega, -1/\omega, 1/(1 - \omega), \omega/(1 + \omega), (\omega - 1)/\omega, 2/(1 - \omega), (\omega - 1)/(1 + \omega)$ and $(1 + \omega)/2$. The region E is shown in Figure 4.6 (b). It is shown by Vulakh [61] that E is the fundamental domain for a normal subgroup Θ_B of $\tilde{\Gamma}_B$, and the images of E under Θ_B tessellate \mathbb{H}^3 by ideal hyperbolic cuboctahedra. The skeleton of this tessellation is a connected graph, which we call a *Farey graph* and denote by \mathcal{F}_B . Part of \mathcal{F}_B is shown in Figure 4.7.

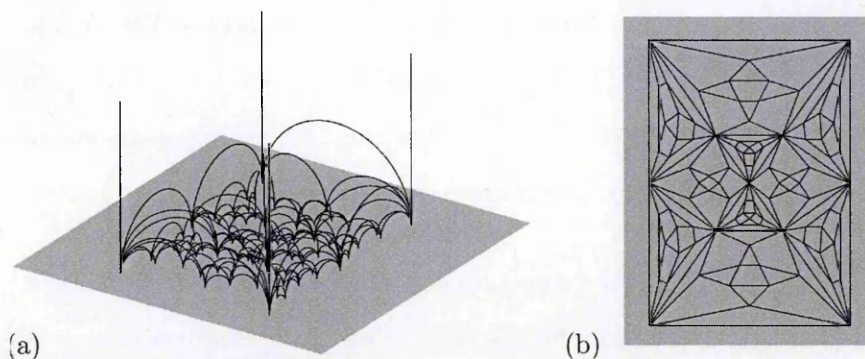


Figure 4.7: (a) Part of the Farey graph \mathcal{F}_B (b) The projection of part of the Farey graph \mathcal{F}_B onto \mathbb{C}

As in [Section 4.1](#), we may alternatively define \mathcal{F}_B as the orbit under $\tilde{\Gamma}_B$ of the hyperbolic geodesic δ between ∞ and 0 in \mathbb{H}^3 . We see that the set of vertices of \mathcal{F}_B is the orbit of ∞ under $\tilde{\Gamma}_B$, and therefore call the vertices of \mathcal{F}_B ∞ -rationals. The set of ∞ -rationals is the set of all complex numbers of the form $x + y\omega$ where x and y are reduced rationals, together with the point ∞ , which we identify with $1/0$; they form a countable, dense subset of $\hat{\mathbb{C}}$, and are the full set of parabolic fixed points of $\tilde{\Gamma}_B$. We may also use this alternative description of \mathcal{F}_B to determine that the neighbours of ∞ in \mathcal{F}_B are precisely the elements of $\mathbb{Z}[\omega]$, and to show that the automorphism group of \mathcal{F}_B is the group generated by $\tilde{\Gamma}_B$ and the anticonformal map $\kappa(z) = -\bar{z}$.

4.2.2 THE CUTTING SEQUENCE OF A GEODESIC IN \mathbb{H}^3

Consider a geodesic ℓ in the upper half-space \mathbb{H}^3 . Such a geodesic will intersect a bi-infinite sequence of cuboctahedra of the graph \mathcal{F}_B . Recall from [Section 1.2.4](#) that each time a geodesic in the upper half-plane \mathbb{H} intersects a face of the Farey graph \mathcal{F} , we say that it *cuts* that face, and we can describe each cut as either a *left cut* or a *right cut*; this gives rise to a sequence of Ls and Rs, symbolising left cuts and right cuts respectively, called the *cutting sequence* of the geodesic, that

describes the way in which ℓ cuts through the faces of \mathcal{F} . We wish to associate to each geodesic ℓ in \mathbb{H}^3 a cutting sequence with respect to the Farey graph \mathcal{F}_B . In Section 4.2.3 we use the cutting sequence to obtain a Bianchi continued fraction expansion of a complex number.

The nature of \mathcal{F}_B as a three-dimensional graph with cuboctahedra as its tessellating entities means that such a cutting sequence is not as straightforward to describe as in the classical case. When the geodesic ℓ intersects, or *cuts*, a cuboctahedron P , it may enter and leave at at one of the fourteen faces of P , or at any one of the 24 edges of P . Furthermore, simply describing the position of the entry and exit faces or edges relative to one another does not uniquely determine the faces or edges. In order to associate to the geodesic ℓ a cutting sequence, we 'pull back' the cuboctahedra that ℓ cuts by applying certain elements of $\tilde{\Gamma}_B$ that map them onto the cuboctahedron E pictured in Figure 4.6 (b), allowing us to classify the cut without ambiguity. Before describing this method in full detail, we introduce some terminology that will be important for describing cuboctahedra in \mathcal{F}_B .

Recall that E denotes the closed ideal cuboctahedron with vertices $\infty, 0, 1, 1 + \omega, \omega, -1/\omega, 1/(1 - \omega), \omega/(1 + \omega), (\omega - 1)/\omega, 2/(1 - \omega), (\omega - 1)/(1 + \omega)$ and $(1 + \omega)/2$, as illustrated in Figure 4.6. We call the four vertical faces of E , together with their vertical edges, the *walls* of E ; the remaining faces, together with their edges, are the *floor* of E . We label the faces of the floor of E T_1, \dots, T_6 and Q_1, \dots, Q_4 as in Figure 4.8.

We let F denote the region that is the union of all images of E under translates from $\tilde{\Gamma}_B$. The region F is convex, and meets the ideal boundary $\hat{\mathbb{C}}$ only at a discrete set of ∞ -rational points. We define the *floor* of F to be the union of all images of the floor of E under the translates from $\tilde{\Gamma}_B$.

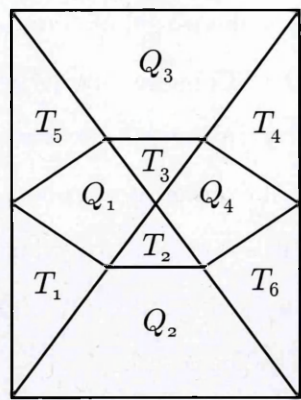


Figure 4.8: The faces of the floor of E

Let ℓ be a directed geodesic in \mathbb{H}^3 that intersects the walls of E, emanating from the inside of E outwards, and which meets the complex plane at a complex number x . We will assume first that x is not an ∞ -rational. Upon leaving E, ℓ passes through an infinite sequence of cuboctahedra E_1, E_2, \dots of \mathcal{F}_B . The first of these, E_1 , is the image of E under one of the translates $\tau_1, \tau_\omega, \tau_1^{-1}$ or τ_ω^{-1} , and hence lies within the region F. Furthermore, there is some k for which E_j lies in F for all $j = 1, \dots, k$, and E_j lies outside F for all $j > k$. In short, upon leaving E, ℓ travels through a number of cuboctahedra of \mathcal{F}_B in F, until it passes through E_k , after which it enters a cuboctahedron E_{k+1} that does not lie in F. This is illustrated in [Figure 4.9](#)

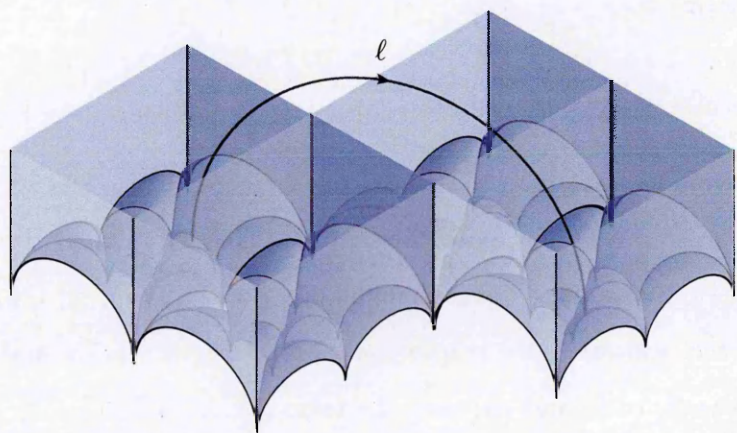


Figure 4.9: The geodesic ℓ passes through a finite number of cuboctahedra with ∞ a vertex

The cuboctahedron E_k is the image of E under the translate $\tau_1^a \tau_\omega^b$ for some integers a and b . Consider the hyperbolic geodesic $\ell' = \tau_1^{-a} \tau_\omega^{-b}(\ell)$. Since ℓ passes through E_k , leaving it to enter E_{k+1} , ℓ' passes through E , leaving it to enter some cuboctahedron $\tau_1^{-a} \tau_\omega^{-b}(E_{k+1})$. Since E_{k+1} does not lie in F , and F is invariant under the translates from Γ_B , ℓ' cannot exit the cuboctahedron E through its walls, and hence must exit through the floor of E .

We now apply a transformation ϕ that is one of four maps ϕ_1 , ϕ_2 , ϕ_3 and ϕ_4 that we will define shortly. The map ϕ fixes E and maps $\tau_1^{-a} \tau_\omega^{-b}(E_{k+1})$ inside F . We can then write the cuboctahedron E_k as $\tau_1^a \tau_\omega^b \phi(E)$. Furthermore, $\phi \tau_1^{-a} \tau_\omega^{-b}(\ell)$ is a directed geodesic in \mathbb{H}^3 that intersects the walls of E , emanating from the inside of E outwards, meeting the complex plane at the complex number x_1 , and so we may repeat the process described above.

We define the map ϕ as follows. Consider the following four Möbius transformations

$$\phi_1(z) = \frac{z}{(1-\omega)z-1};$$

$$\phi_2(z) = \frac{2z-(1+\omega)}{(1-\omega)z-2};$$

$$\phi_3(z) = \frac{(\omega-1)z+2}{(1+\omega)z+(1-\omega)};$$

$$\phi_4(z) = \frac{\omega z - (1-\omega)}{(1+\omega)z - \omega}.$$

The involutions ϕ_i for $i = 1, 2, 3, 4$ are elements of $\tilde{\Gamma}_B$ that fix the cuboctahedra E . Furthermore, the map ϕ_1 takes the faces T_1 , T_2 , Q_1 and Q_2 to the walls of E ; the map ϕ_2 takes the faces T_3 , T_4 , Q_3 and Q_4 to the walls of E ; the map ϕ_3 takes the faces T_3 , T_5 , Q_1 and Q_3 to the walls of E ; and the map ϕ_4 takes the faces T_2 , T_6 , Q_2 and Q_4 to the walls of E . We subdivide the floor of E as follows. We let $R_1 = \overline{T_1 \cup T_2 \cup Q_1 \cup Q_2}$ where \overline{S} is denotes the closure of a set

S. We let $R_2 = \overline{T_3 \cup T_4 \cup Q_3 \cup Q_4} \setminus R_1$, $R_3 = \overline{T_4 \setminus \{R_1 \cup R_2\}}$, and $R_4 = \overline{T_6 \setminus \{R_1 \cup R_2\}}$. These regions are illustrated in Figure 4.10.

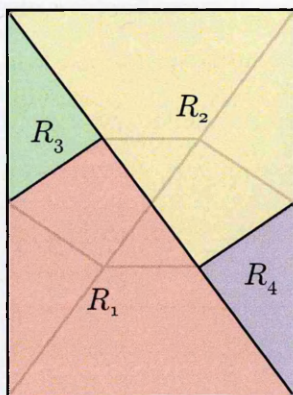


Figure 4.10: The faces of the floor of E

If ℓ' intersects the floor of E in the region R_j then we let $\phi = \phi_j$. Let $\ell_1 = \phi\ell' = \phi\tau_1^{-a}\tau_\omega^{-b}(\ell)$.

Lemma 4.7. *The geodesic ℓ_1 is a directed geodesic in \mathbb{H}^3 that intersects the wall of E , emanating from the inside of E outwards, and which meets the complex plane at a complex number that is not an ∞ -rational.*

Proof. The map $\phi = \phi_j$ is chosen so that the point at which ℓ' intersects the floor of E is mapped into the walls of E . The result follows. \square

We can therefore repeat the process described in this section indefinitely, producing a sequence of Möbius transformations f_k of the form

$$f_k = \tau_1^{a_k} \tau_\omega^{b_k} \phi_{j_k}$$

such that the directed geodesic ℓ cuts through the cuboctahedra $E, f_1(E), f_1 f_2(E), \dots$ in that order. We call the sequence of triples

$$(j_1, a_1, b_1), (j_2, a_2, b_2), \dots$$

the *cutting sequence* of ℓ with respect to \mathcal{F}_B .

We now consider the case in which the endpoint x of the vertex ℓ is ∞ -rational. In this case, we may produce a cutting sequence following precisely the same procedure as in the case when x is not ∞ -rational. In this case, however, we may come to a point where x is a vertex of one of the cuboctahedra $f_1 \cdots f_n(E)$, and the procedure terminates. We then obtain a finite cutting sequence $(j_1, a_1, b_1), \dots, (j_n, a_n, b_n)$.

4.2.3 THE BIANCHI CONTINUED FRACTION EXPANSION OF A COMPLEX NUMBER

We will now use the cutting sequence process described in Section 4.2.2 to obtain a Bianchi continued fraction expansion of a complex number x . We begin by considering the case in which x is not an ∞ -rational. Let δ_x denote the geodesic between ∞ and x , directed from ∞ towards x . Although δ_x does not emanate from inside the cuboctahedron E , it is clear that there are integers a_1 and b_1 such that $\tau_1^{-a_1} \tau_\omega^{-b_1}(\delta_x)$ intersects the floor of the cuboctahedron E . It follows that we may apply the procedure described in Section 4.2.2 to obtain a cutting sequence $(j_1, a_1, b_1), (j_2, a_2, b_2), \dots$ of δ_x . Write $f_k = \tau_1^{a_k} \tau_\omega^{b_k} \phi_{j_k}$ and $F_n = f_1 \cdots f_n$. Then the geodesic δ_x intersects each of the cuboctahedra $F_n(E)$ for $n = 1, 2, \dots$.

Theorem 4.8. *Let y be a vertex of E . Then $F_n(y) \rightarrow x$ as $n \rightarrow \infty$.*

Proof. This theorem follows from [44, Theorem 5]; we sketch a proof here. For each n , let E_n denote the cuboctahedron $F_n(E)$. There is some face F_n of E_n such that every vertex of E_n lies on or within the Euclidean circle \mathcal{C}_n in $\hat{\mathbb{C}}$ that circumscribes F_n . In fact, the circles \mathcal{C}_n form a nested chain of circles whose radii converge to zero, all of which contain x . It follows that $F_n(y)$ converges as $n \rightarrow \infty$, and that $F_n(y)$ must converge to x . \square

We show next that the maps F_n give rise to an infinite Bianchi continued fraction of x . From now on, given a complex number $z = x + y\omega$ with x and y integers, we shall write t_z for the map $\tau_1^x \tau_\omega^y \sigma$. We can write the maps ϕ_j for $j = 1, \dots, 4$ as follows:

$$\phi_1(z) = t_0 t_{1-\omega} t_0 \sigma';$$

$$\phi_2(z) = t_\omega t_{-1-\omega} t_\omega t_1 \sigma';$$

$$\phi_3(z) = t_0 t_{\omega-1} t_{-1-\omega} t_\omega t_1 \sigma';$$

$$\phi_4(z) = t_0 t_{-1} t_\omega t_1 \sigma'.$$

Consider the map $F_n = f_1 \cdots f_n$. By using relations of the group $\tilde{\Gamma}_B$, and in particular the relation

$$\sigma' \tau_1^x \tau_\omega^y = \sigma \tau_1^{-x} \tau_\omega^{-y} \sigma \sigma',$$

we can write

$$F_n = t_{n_1} \cdots t_{n_m} \sigma',$$

where $n_j \neq 0$ for $j = 2, \dots, m-1$. Furthermore, it can be shown that

$$F_{n+1} = t_{(n+1)_1} \cdots t_{(n+1)_l} \sigma',$$

where at least the first $m-2$ terms $(n+1)_j$ are equal to n_j . Therefore we can find an infinite sequence of maps t_{b_1}, t_{b_2}, \dots such that if $T_n = t_{b_1} \cdots t_{b_n}$, $F_n = T_n g_n$ where $g_n(\infty)$ lies in the unit disc in \mathbb{C} . It follows that

$$\lim_{n \rightarrow \infty} T_n(\infty) = \lim_{n \rightarrow \infty} F_n(\infty) = x,$$

and so the maps F_n give rise to the Bianchi continued fraction expansion $[b_1, b_2, \dots]$ of x .

We end this section by considering the case in which x is an ∞ -rational. The procedure explained in Section 4.2.2 gives rise to a finite sequence of maps $F_1 \dots F_n$ such that x is a vertex of $F_n(E)$. Writing

$$F_n = t_{b_1} \dots t_{b_m} \sigma',$$

we see that $[b_1, \dots, b_m]$ is a Bianchi continued fraction expansion of the vertex $F_n(0)$ of $F_n(E)$. However, it may be the case that x is not the vertex $F_n(0)$, say $x = F_n g(0)$ where g is an element of $\tilde{\Gamma}_B$ fixing E . In this situation, we can obtain a finite Bianchi continued fraction expansion of x by writing g in terms of the maps t_z and σ' .

4.3 SEMIGROUPS OF MÖBIUS TRANSFORMATIONS

So far, this thesis has focused on ways of using hyperbolic geometry to study classes of continued fractions that can be seen as generalisations of the classical regular continued fractions. Many other attempts to generalise the theory of regular continued fractions may be found in the literature, from higher-dimensional analogues to continued fractions whose coefficients belong to more general number systems, such as the set of quaternions. In this section we look at an altogether different way of generalising regular continued fractions.

Consider the classical regular continued fraction

$$[b_1, b_2, \dots] = b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \dots}},$$

where each b_i is an integer, and $b_i > 0$ for $i > 1$. In Section 1.2.1, it is noted that a regular continued fraction may be viewed as a sequence of Möbius transformations belonging to the modular group Γ . We will now examine a slightly different connection between regular continued fractions and Möbius transformations. Consider the two Möbius transformations

$$f(z) = \frac{z}{z+1} = \frac{1}{1+1/z} \quad \text{and} \quad g(z) = z+1.$$

The maps f and g generate the modular group Γ . Given a simple continued fraction $[b_1, b_2, \dots]$ we can write the convergents $c_k = [b_1, \dots, b_k]$ as

$$c_k = g^{b_1} f^{b_2} g^{b_3} \dots h^{b_k}(\infty), \quad (4.2)$$

where the map h equals f if k is even, and g if k is odd. Since f and g generate Γ , any element of Γ may be written in the form given in Equation (4.2). Not all such elements, however, will correspond to a regular continued fraction; it is precisely those in which b_2, \dots, b_k are positive – those that lie in the semigroup generated by f and g – that correspond to regular continued fractions. Our idea is, therefore, to generalise regular continued fractions by considering two-generator semigroups of (real) Möbius transformations. Before explaining this generalisation in more detail, we introduce the relevant background material on the theory of semigroups.

4.3.1 SEMIGROUPS OF MÖBIUS TRANSFORMATIONS

Let M denote the set of all conformal real Möbius transformations, that is, the set of Möbius transformations of the form

$$f(z) = \frac{az + b}{cz + d},$$

such that a, b, c and d are real numbers and $ad - bc > 0$. Such Möbius transformations act on the upper half-plane \mathbb{H} . We consider a *semigroup* of Möbius transformations S to be any subset of M that is closed under composition. Given a finite subset F of M , the set of all compositions $f_1 \cdots f_n$ where $n \geq 1$ and $f_i \in F$ for each $i = 1, \dots, n$ forms a semigroup S_F ; we call this semigroup the *semigroup generated by F* .

Given a finite subset F of M we define a *composition sequence* generated by F to be a sequence $F_k = f_1 \cdots f_k$ where $f_i \in F$ for each $i = 1, \dots, k$. Notice that the set of all composition sequences generated by F is in bijection with the set of elements of the semigroup S_F generated by F . We say that a composition sequence *converges ideally* to a point x in $\hat{\mathbb{R}}$ if there is some point ζ in \mathbb{H} such that $F_k(\zeta) \rightarrow x$ as $k \rightarrow \infty$. Here, convergence is in the spherical metric on \mathbb{H} . Note that the definition is independent of the choice of ζ .

We define the *forward limit set* $\Lambda^+(S_F)$ of S_F to be the set of accumulation points in $\hat{\mathbb{R}}$ of the set $\{f(\zeta) \mid f \in S_F\}$ for some $\zeta \in \mathbb{H}$. Again, convergence is in the spherical metric on \mathbb{H} , and the definition is independent of the choice of ζ . We may also define the *backward limit set* $\Lambda^-(S_F)$ of S_F to be the set of accumulation points in $\hat{\mathbb{R}}$ of the set $\{f^{-1}(\zeta) \mid f \in S_F\}$. For the semigroups that we are interested in, the intersection $\Lambda^+(S_F) \cap \Lambda^-(S_F)$ of the forward and backward limit sets is empty. This implies firstly that every composition sequence gen-

erated by F converges ideally, and secondly that the forward limit set of S_F is precisely the set of limit points of composition sequences generated by F .

4.3.2 S -CONTINUED FRACTIONS

Let f and g be two conformal real Möbius transformations that are either parabolic or hyperbolic, and let S be the semigroup generated by the set $\{f, g\}$. Let a and b be the attracting fixed points of f and g respectively (if f is parabolic then a is the unique fixed point of f , and similarly for b and g). We will assume henceforth that $-\infty < b \leq a < \infty$, and that both f and g map the interval $I = [a, b]$ strictly within itself. Notice that for any composition sequence F_1, F_2, \dots generated by $\{f, g\}$,

$$I \supset F_1(I) \supset F_2(I) \supset \dots$$

It can be shown that any such composition sequence converges ideally to a point x in I ; furthermore, the composition sequence converges to x locally uniformly on the interior of I .

Let x denote the limit point of a composition sequence $f^{b_1} g^{b_2} \dots$. We write $x = [0, b_1, b_2, \dots]$. If x is the limit point of a composition sequence $g^{b_0} f^{b_1} \dots$. Then we write $x = [b_0, b_1, \dots]$. In summary, the limit point of any composition sequence can be represented by $[b_0, b_1, b_2, \dots]$, where b_0 is zero or a positive integer, and each other b_i is a positive integer. We call $[b_0, b_1, b_2, \dots]$ the *S-continued fraction expansion* of x . Let $R(S)$ denote the set of images of the points a and b under elements of S . We call points in $R(S)$ *S-rationals*. Every point in $\Lambda^+(S_F) \setminus R(S)$ can be represented by an *S-continued fraction*, although this representation may not be unique.

The naming of these composition sequences as S -continued fractions is motivated by the following example. Let

$$f(z) = \frac{z}{z+1} = \frac{1}{1+1/z} \quad \text{and} \quad g(z) = z+1.$$

The maps f and g are parabolic conformal Möbius transformations with f fixing 0 and g fixing ∞ . The semigroup S generated by $\{f, g\}$ has forward limit set $[0, \infty]$, and the set $R(S)$ is the set of all rational numbers in the interval $[0, \infty]$ and the point ∞ . By the above discussion, any rational number x in $[0, \infty]$ may be represented by an S -continued fraction $[b_0, b_1, \dots]$. Furthermore, $[b_0, b_1, \dots]$ is simply the sequence of Möbius transformations F_1, F_2, \dots such that the points $F_1(\infty), F_2(\infty), \dots$ are the consecutive convergents of the regular continued fraction expansion of x . In this sense, S -continued fractions are a natural generalisation of regular continued fractions. In the remainder of this section we explore some topics in the classical theory of regular continued fractions, with the aim of generalising these results to S -continued fractions.

The behaviour of S -continued fractions is determined largely by the behaviour of the maps f and g , which can be classified into one of three categories depending on the order in which the points $f(b)$ and $g(a)$ lie on \mathbb{R} . We have three possibilities: $f(b) < g(a)$; $f(b) > g(a)$; $f(b) = g(a)$. These three cases are illustrated in Figure 4.11.

It can be shown that $\Lambda^+(S) = I$ if and only if $f(b) \geq g(a)$. It follows that there is an S -continued fraction expansion of every real number in $\Lambda^+(S) \setminus R(S)$ if and only if $f(b) \geq g(a)$. On the other hand, only when $f(b) \leq g(a)$ is there a unique S -continued fraction expansion of every real number $\Lambda^+(S) \setminus R(S)$. It is only when $f(b) = g(a)$, therefore, that every real number in $\Lambda^+(S) \setminus R(S)$ has a unique S -continued fraction expansion.

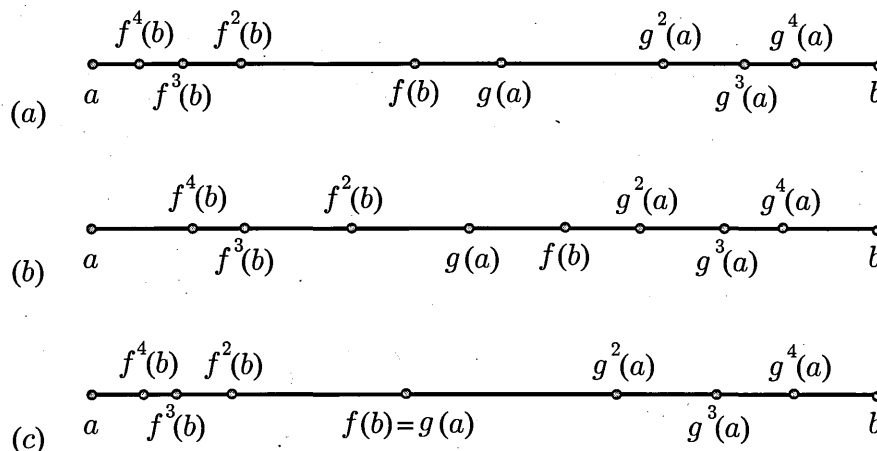


Figure 4.11: (a) $f(b) < g(a)$ (b) $f(b) > g(a)$ (c) $f(b) = g(a)$

From now on, we shall assume that $f(b) = g(a) = c$ for some real number $c \in I$. We begin by showing that the collection of all such pairs $\{f, g\}$, up to conjugation, forms a two-parameter family that can be determined by the multipliers of f and g . Consider real numbers a, b and c , with $a < c < b \leq \infty$, and write $c' = (c - a)/(c - b)$. Given real numbers $0 < m_f, m_g \leq 1$, there are unique maps

$$f(z) = \frac{(a - bc' + ac'/m_f)z - (a^2 - abc' + abc'/m_f)}{(1 - c' + c'/m_f)z + (ac' - a - bc'/m_f)},$$

and

$$g(z) = \frac{(a - bc' - b/m_g)z + ab/m_g + b^2c' - ab}{(1 - c' - 1/m_g)z + a/m_g + bc' - b},$$

such that a is the attracting fixed point of f , b is the attracting fixed point of g , and $f(b) = g(a) = c$; the real numbers m_f and m_g are the multipliers of f and g respectively. There is a unique Möbius transformation mapping a to 0, b to ∞ and c to 1; conjugating by this map gives us

$$f(z) = \frac{z}{1/m_f + z} \quad \text{and} \quad g(z) = z/m_g + 1.$$

Therefore, up to conjugation, we have only a two-parameter family of maps to consider, determined by the multipliers m_f and m_g of the maps f and g . In the special case $m_f = m_g = 1$, the maps f and g give rise to regular continued fractions.

Given $0 < m_f, m_g \leq 1$, let

$$f(z) = \frac{z}{1/m_f + z} \quad \text{and} \quad g(z) = z/m_g + 1.$$

Denote by S the semigroup generated by the set $\{f, g\}$. The point 0 is the attracting fixed point of f , the point ∞ is the attracting fixed point of g , and $f(\infty) = g(0) = 1$. The action of the maps f and g on the interval $[0, \infty]$ is illustrated in Figure 4.12

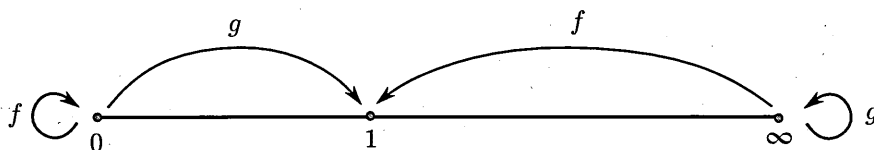


Figure 4.12: The action of the maps f and g on $[0, \infty]$

We may define a Farey graph \mathcal{F}_S in \mathbb{H} as follows. Let δ denote the hyperbolic geodesic between 0 and ∞ . Then the graph \mathcal{F}_S is the orbit of δ under S . The vertices of the graph are precisely the S -rationals. Two vertices are joined by an edge in \mathcal{F}_S if and only if they are the endpoints of a common interval $h(I)$ for some h in S . In the special case where f and g are such that the multipliers m_f and m_g are both 1 , we obtain the classical Farey graph \mathcal{F} , as illustrated in Figure 1.1 (although only the portion of \mathcal{F} with vertices in the interval $[0, \infty]$ is included). In all other cases, the graph \mathcal{F}_S is isomorphic as a graph to \mathcal{F} . This is a likely indicator that the theory of S -continued fractions will have parallels with the theory of regular continued fractions.

Given a point x in I that is not an S -rational, we may construct an S -continued fraction expansion of x as follows. Firstly, notice that x lies inside the interval $g^{b_0}(I)$ for some maximal $b_0 = 0, 1, \dots$; see

Figure 4.13. Now define $x_1 = g^{-b_0}(x)$. Since x lies between $g^{b_0}(0)$ and $g^{b_0+1}(0)$, x_1 lies between 0 and $g(0) = f(b)$. Now define b_1 to be the largest integer such that x_1 lies inside $f^{b_1}(I)$. Since x_1 lies between 0 and $f(a)$, $b_1 \geq 1$. Now define $x_2 = f^{-b_1}(x_1) = f^{-b_1}g^{-b_0}(x)$. Since x_1 lies between $f^{b_1}(0)$ and $f^{b_1+1}(0)$, x_2 lies between $g(0)$ and ∞ , and we can define b_2 to be the largest integer such that x_1 lies inside $g^{b_2}(I)$. We can continue in this fashion to obtain a sequence of maps $g^{b_0}, f^{b_1}, g^{b_2}, \dots$ such that

$$x = g^{b_0}f^{b_1}g^{b_2} \dots h^{b_k}(x_{k-1}) = F_k(x_{k-1})$$

where $h = f$ if k is even and g if k is odd. Since each x_k lies in I , and any composition sequence generated by $\{f, g\}$ converges locally uniformly on the interior of I , and $x \in F_k(I)$ for each k , $[b_0, b_1, b_2, \dots]$ is an S -continued fraction expansion of x .

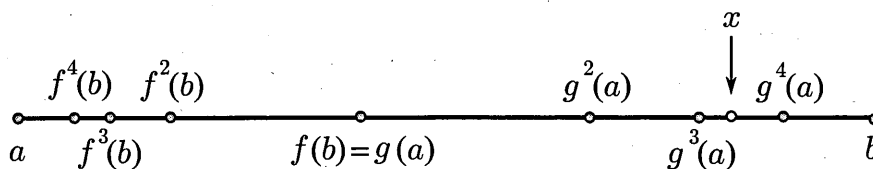


Figure 4.13: The point x lies inside $g^{b_1}(I)$

When x is an S -rational, we may apply the same procedure as when x is not S -rational, except that eventually x_k will be equal to $f^{b_k}(0)$ (or $g^{b_k}(0)$), and the algorithm terminates. Then we obtain a finite S -continued fraction expansion $[b_0, \dots, b_k]$ of x .

It can be shown that the sequence of edges of \mathcal{F}_S given by $F_k(\delta)$ for $k = 1, 2, \dots$ forms an infinite (or finite) path in \mathcal{F}_S from ∞ to x . In the case in which $\mathcal{F}_S = \mathcal{F}$, this path coincides with the path of convergents of the regular continued fraction expansion of x .

This algorithm establishes the existence and uniqueness of an S -continued fraction expansion of any non-negative real number. In Section 4.3.4 we consider other properties of regular continued fractions

that may hold for S -continued fractions, with a view to building up a theory of S -continued fractions analogous to the theory of regular continued fractions.

4.3.3 DIOPHANTINE APPROXIMATION

One of the principle uses of regular continued fractions is in the field of Diophantine approximation, as discussed in Section 3.3. Consider the maps

$$f(z) = \frac{z}{z+1} = \frac{1}{1+1/z} \quad \text{and} \quad g(z) = z+1.$$

Recall that the maps f and g are parabolic conformal Möbius transformations with f fixing 0 and g fixing ∞ . The semigroup S generated by $\{f, g\}$ gives rise to the classical regular continued fractions. It has forward limit set $\Lambda^+(S) = [0, \infty]$, and the set of S -rational numbers $R(S)$ is the set of all rational numbers in the interval $[0, \infty]$. Classical Diophantine approximation looks at the approximation of irrational numbers by rational numbers, or, in other words, the approximation of points in $\Lambda^+(S) \setminus R(S)$ by points in $R(S)$.

Consider a semigroup S_F , as defined in Section 4.3.1, that is generated by two maps f and g with attracting fixed points a and b respectively, and $f(b) = g(a) = c$. We may generalise the notion of Diophantine approximation by look more generally at the approximation of points in $\Lambda^+(S_F) \setminus R(S_F)$ by points in $R(S_F)$. Although there is a wealth of literature on the approximation of points in the limit set of a Fuchsian group (beginning, for example, with works such as [40] and [58]), there seems to be no such research into the analogous problem for semigroups. In this section we look at how results such as those obtained in Section 3.3 might adapt for the approximation of points

in $\Lambda^+(S_F) \setminus R(S_F)$ by points in $R(S_F)$, and we begin by generalising the notion of Ford circles.

Recall that a Ford circle C_x is the horocycle based at x with Euclidean radius $\text{rad}[C_x] = 1/(2b^2)$. The Ford circle C_∞ is the line $\text{Im}(z) = 1$ in \mathbb{C} together with the point ∞ . Note that the rational numbers are the image of ∞ under the modular group Γ . Correspondingly, the collection of all Ford circles is the image of C_∞ under Γ . The fact that the collection of all Ford circles is invariant under Γ follows from the fact that the multiplier of both generating maps f and g is 1. No collection of horocycles in \mathbb{H} will be invariant under the action of the semigroup S_F when one of the multipliers is not equal to 1; this causes some difficulty when working with Ford circles in this setting.

We will now show how to construct the family of horocycles that we call *Ford S_F -circles*. We will assume, without loss of generality, that the generators f and g of S_F have been conjugated so that

$$f(z) = \frac{z}{1/m_f + z} \quad \text{and} \quad g(z) = z/m_g + 1.$$

Every S_F -rational is therefore the image of either 0 or ∞ under S_F . We begin by placing two initial horocycles, C_0 and C_∞ , at the points 0 and ∞ ; all others may be produced by taking the images of C_0 and C_∞ under S_F . We choose the radii of C_0 and C_∞ so that $f(C_\infty) = g(C_0)$ and the three horocycles C_0 , C_∞ and $f(C_\infty)$ are pairwise tangent. This is the case if and only if the radius r_0 of C_0 and the height h_∞ of C_∞ are

$$r_0 = \sqrt{\frac{f'(b)b^2}{4g'(0)}} \quad \text{and} \quad h_\infty = \sqrt{\frac{f'(b)b^2}{g'(0)}}.$$

The images of these two horocycles under S_F give rise to an infinite collection of horocycles based at each S_F -rational number (providing that m_f and m_g are not both 1). Examples of the collection of Ford

S_F -circles for which $m_f = m_g = 0.5$, $m_f = 0.5$ and $m_g = 0.2$ and $m_f = 0.5$ and $m_g = 0.8$ are shown in Figure 4.14, Figure 4.15 and Figure 4.16 respectively. We define the Ford S_F -circle based at an S_F -rational number x to be the largest of the horocycles based at x .

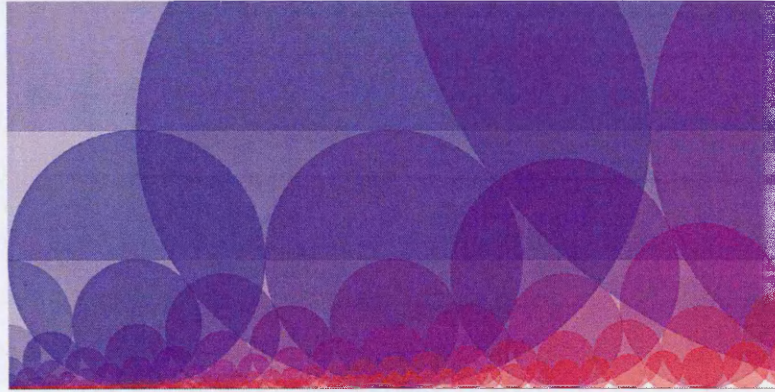


Figure 4.14: Part of the collection of S_F -Ford circles when S_F is generated by $\{f, g\}$ with multipliers $m_f = 0.5$ and $m_g = 0.5$

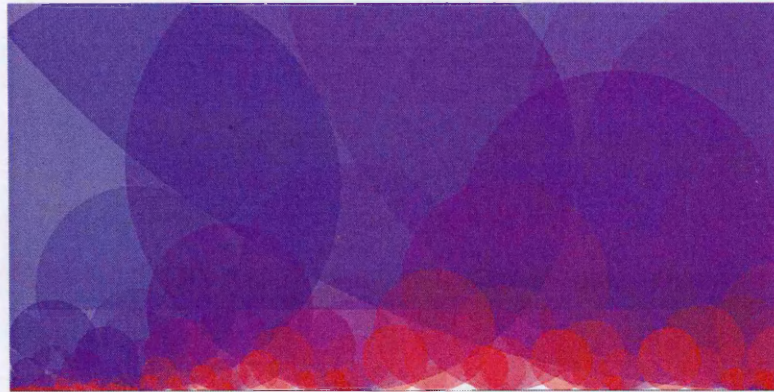


Figure 4.15: Part of the collection of S_F -Ford circles when S_F is generated by $\{f, g\}$ with multipliers $m_f = 0.5$ and $m_g = 0.2$

We will look briefly at several ways in which Ford S_F -circles may be used to study the approximation of points in $\Lambda^+(S_F) \setminus R(S)$ by points in $R(S)$.

As in Section 3.3.2, for each S_F -rational number u , we define a function R_u such that for any real number x , $R_u(x)$ is the Euclidean radius of the horocycle based at x that is externally tangent to C_u ; this

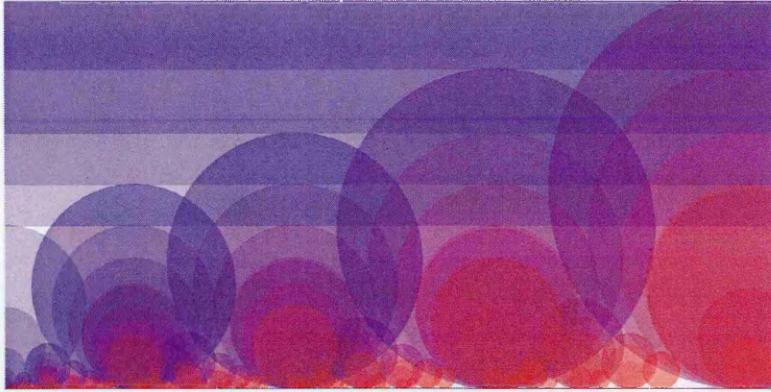


Figure 4.16: Part of the collection of S_F -Ford circles when S_F is generated by $\{f, g\}$ with multipliers $m_f = 0.3$ and $m_g = 0.8$

is illustrated in Figure 3.21. We define a strong S_F -approximant of a point in $\Lambda^+(S_F) \setminus \mathbb{R}(S)$ to be an S_F -rational u with the property that for each S_F -rational w with $\text{rad}[C_u] \leq \text{rad}[C_w]$, we have $R_u(x) \leq R_w(x)$, with equality if and only if $w = u$.

Conjecture. *An S_F -rational is a strong S_F -approximant of an S_F -irrational number x if and only if it is a convergent of the S_F -continued fraction of x .*

It may also be possible to use Ford S_F -circles to define and find a Hurwitz constant for approximation by S_F -rationals, as in Section 3.3.3. This leads to the subject of badly approximable numbers. A point $x \in \mathbb{R} \setminus \mathbb{Q}$ is *badly approximable* if for any rational p/q ,

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}$$

for some uniform constant c . Recall that in Section 3.3.3, generalised Ford circles, called Ford k -circles, were introduced: the *Ford k -circle* based at a point x is obtained by scaling the Ford circle based at x by a factor of k . Then a point x is badly approximable if and only if there is some k such that the vertical line δ_x from x to ∞ intersects no Ford k -circle.

Just as we can introduce Ford k -circles, we can also introduce *Ford S_F - k -circles*, which we obtain from Ford S_F -circles by uniformly scaling them by a factor of k . Then we can define a *badly S_F -approximable* number to be an S_F -irrational number x such that the vertical line δ_x from x to ∞ intersects no Ford S_F - k -circle for S_F . We end this section with the following conjecture.

Conjecture. *An S_F -irrational number x is badly S_F -approximable if and only if $x = [b_0, b_1, \dots]$ where the set $\{b_i \mid i = 0, 1, \dots\}$ is bounded above.*

4.3.4 AREAS FOR FURTHER RESEARCH

The generalisation of regular continued fractions to S -continued fractions introduced in this section is new and is hence relatively undeveloped. There seems to be, however, potential for a rich theory of S -continued fractions to be developed. In this section we outline some areas for further study that may lead to new and interesting results. Throughout, we will consider S_F to be a semigroup, as defined in Section 4.3.1, that is generated by two maps f and g with attracting fixed points a and b respectively, with $a < b$ and both $f([a, b]) \subset [a, b]$ and $g([a, b]) \subset [a, b]$.

A theory of S_F -continued fractions

The techniques discussed previously in this section, namely the Farey graph \mathcal{F}_{S_F} and Ford S_F -circles, were introduced with the aim of building up a theory of S_F -continued fractions. The existence and uniqueness of the S_F -continued fraction expansion of a point in $\Lambda^+(S_F)$ was discussed, as were possible results in Diophantine approximation. The structure of the Farey graph S_F suggests that other classical results in the theory of regular continued fractions might hold for S_F -continued fractions.

We begin by considering the well-known theorem of Lagrange regarding periodic regular continued fractions (see Section 3.2.5). We say that an S_F -continued fraction is periodic if it is of the form

$$x = [b_1, b_2, \dots, b_n, \overline{a_1, a_2, \dots, a_m}],$$

where the string a_1, \dots, a_m repeats infinitely. We say that the expansion is *purely periodic* if the length of the sequence b_1, \dots, b_n has length zero. Let $A(S_F)$ denote the set of attracting fixed points of S_F .

Conjecture. *The set of attracting fixed points $A(S_F)$ of elements of S_F is equal to the set of points x in I that have purely periodic S_F -continued fraction expansions.*

Corollary 4.9. *The set of $S_F(A(S_F))$ is equal to the set of points x in I that have periodic S_F -continued fraction expansions.*

The motivation behind this conjecture is as follows. Lagrange's theorem states that the regular continued fraction expansion of an irrational number x is periodic if and only if x is a quadratic irrational. The set of all quadratic irrationals is, we believe, also the set $S_F(A(S_F))$ the semigroup S_F that gives rise to regular continued fractions (see also Theorem 3.14).

It is likely that other classical results from the theory of continued fractions, such as Serret's theorem (see, for example, Section 3.2.4), will also have analogues for S_F -continued fractions. Given a point x that is not S_F -rational, we say that a point y is S_F -equivalent to x if there is some element h of S_F such that $y = h(x)$.

Conjecture. *Given a point x that is not S_F -rational, a point y is S_F -equivalent to x if and only if there is a positive integer k such that the S_F -continued fraction expansions of x and y ,*

$$x = [a_1, a_2, \dots] \quad \text{and} \quad y = [b_1, b_2, \dots],$$

satisfy $a_i = b_{k+i}$ for all $i = 2, 3, \dots$

The cases $f(b) < g(a)$ and $f(b) > g(a)$

So far, this section has focused on S_F -continued fractions where the generators f and g of S_F satisfy $f(b) = g(a) = c$ for some $a < c < b$. We now briefly consider the two cases when $f(b) < g(a)$ and $f(b) > g(a)$.

The case $f(b) < g(a)$ is relatively simple. Every point in the set $\Lambda^+(S_F) \setminus R(S_F)$ has a unique S_F -continued fraction expansion, but now $[a, b]$ lies strictly inside in $\Lambda^+(S_F)$; in fact, $\Lambda^+(S)$ is a Cantor set. In order to determine precisely which real numbers can be represented by an S_F -continued fraction, it is first necessary to determine which points lie in $\Lambda^+(S_F)$. As a starting point, it might be possible to use results from the literature to compute the Hausdorff dimension of these limit sets.

In the case $f(b) > g(a)$, the two intervals $f([a, b])$ and $g([a, b])$ overlap. This means that every point in $\Lambda^+(S_F)$ can be represented by an S_F -continued fraction, but this expansion may not be unique. When applying the algorithm described in Section 4.3.2, a point may lie in two intervals, in which case we have to make a choice as to which interval we use before we can continue the algorithm. An idea would be to use a *choice function*, as in the work of Dani and Nogueira [11] (see Section 4.1.3), to associate an interval to each point. Given a choice function, a unique S_F -continued fraction expansion of every point in $[a, b]$ would be obtained, and it may be possible to study such continued fractions exactly as in the case in which $f(b) = g(a)$. Another direction to explore in the case $f(b) > g(a)$ would be the number of S_F -continued fraction expansions of a given point in $[a, b]$.

Other areas for further research

The semigroups S_F introduced thus far are very special examples within the much more general class of two-generator semigroups of Möbius transformations. It would be interesting to see whether or not a similar theory of S -continued fractions can be developed when S is a more general semigroup. One possibility would be to investigate the situation where the maps f and g that generate S don't map the interval $[a, b]$ strictly within itself. This happens when f and g are hyperbolic and their axes point in different directions, as illustrated in Figure 4.17.

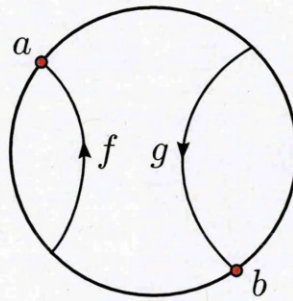


Figure 4.17: The axes of the hyperbolic Möbius transformations f and g point in opposite directions

Another possibility would be to consider the case in which S is generated by three, or more, Möbius transformations, or when the Möbius transformations f and g have complex coefficients. These questions might lead to a more general study of the theory of semigroups of Möbius transformations, a relatively young but fertile area.

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